

Modes of Convergence to the Truth: Steps toward a Better Epistemology of Induction

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April 26, 2018

Abstract

There is a large group of people engaging in normative or evaluative studies of induction, such as computer scientists, statisticians, Bayesians, learning theorists, and other formal epistemologists. They have provided many positive results for justifying (to a certain extent) various kinds of inductive inferences, but they all have said little about a very familiar kind. I call it *full* enumerative induction, of which an instance is this: “We’ve seen this many ravens and they all are black, so all ravens are black”—without a stronger premise such as IID or a weaker conclusion such as “all the ravens observed in the future will be black”. To remedy this, I propose that Bayesians be *learning-theoretic* and learning-theorists be *truly* learning-theoretic—in two steps. (i) Understand certain modes of convergence to the truth as epistemic ideals. (ii) See whether full enumerative induction can be justified as—that is, proved to be—a necessary feature of every learning method that achieves the highest epistemic ideal achievable for tackling the problem of whether all ravens are black. The answer is positive, thanks to a new result in learning theory (whose Bayesian version is proved as well). The conceptual breakthrough consists in introducing a mode of convergence slightly weaker than Gold’s (1965) and Putnam’s (1965) identification in the limit; I call it *almost everywhere* convergence to the truth, where the conception of “almost everywhere” is borrowed from geometry and topology. (Those who only care about the philosophical ideas rather than the technical details can read just the first two sections, i.e. up to page 16.)

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1 Introduction

The *general* problem of induction is the problem of identifying two things:

- (i) the range of the inductive inferences that we can justify,
- (ii) the extent to which we can justify them.

Under the general problem there are several subproblems. There is, for example, the special subproblem of how it is possible to reliably infer causal structures solely from observational data without experimentation, which has attracted many statisticians, computer scientists, and philosophers.¹ And there is the more general subproblem of whether it is possible to escape Hume's dilemma—a dilemma that aims to undermine any justification of any kind of inductive inference.² In this paper I want to focus on a subproblem of induction that is more *normative and evaluative* in nature.

Here is the background. Normative/evaluative studies of inductive inference are pursued in many mathematical disciplines: formal epistemology, statistics, and theoretical computer science. But somewhat curiously, they all have said little about the most familiar kind of inductive inference, of which an instance is this:

(FULL ENUMERATIVE INDUCTION) We have observed this many ravens and they all are black; so all ravens are black.

To be sure, those disciplines have had much to say about enumerative induction, but typically only about a *restricted* version that weakens the conclusion or strengthens the premise. Here is an example:

(RESTRICTED ENUMERATIVE INDUCTION) We have observed this many ravens and they all are black; so all the ravens *observed in the future* will be black

This is the kind of enumerative induction studied by Bayesian confirmation

¹For a groundbreaking treatment of this problem, see Spirtes, Glymour, and Scheines (1993/[2000]).

²See Hume (1777) for his formulation of the dilemma, and Reichenbach (1938, sec. 38) for a more powerful formulation of the dilemma. A quite comprehensive list of attempted responses is provided in Salmon (1966: chap. 2).

theorists,³ and also by formal/algorithmic learning theorists.⁴ Classical statisticians and statistical learning theorists study an *even more* restricted version that adds a substantial assumption of IID (independent and identically distributed) random variables, where the randomness is due to objective chance.

So they all set aside a serious study of full enumerative induction. But why? The reason for statisticians is obvious: their primary job is to study inductive inferences under the IID assumption or the like. The reason for Bayesians and formal learning theorists is deeper. Let me explain.

It is possible that there are nonblack ravens but an inquirer never observes one throughout her entire life (even if she is immortal). Call such a possibility a *Cartesian scenario of induction*, for it can be (but need not be) realized by a Cartesian-like demon who always hides nonblack ravens from the inquirer's sight. Each Cartesian scenario of induction has a *normal counterpart*, in which the inquirer receives exactly the same data in time (without observing a non-black raven) and, fortunately, all ravens are black. A Cartesian scenario of induction and its normal counterpart are empirically indistinguishable, but in the first it is false that all ravens are black, while in the second it is true. And that causes trouble for both formal learning theorists and Bayesians.

For Bayesians, justification of full enumerative induction requires justifying a prior probability distribution that strongly disfavors a Cartesian scenario of induction and favors its normal counterpart. Carnap and other Bayesian confirmation theorists seem to never mention such an anti-Cartesian prior in their papers, possibly because they cannot justify it or simply because they do not care about it.

A subjective Bayesian would say that such an anti-Cartesian prior is epistemically permissible, but only because she thinks that any probabilistic prior is epistemically permissible—even including those priors that strongly favor a Cartesian scenario of induction and will thereby lead to counterinduction:

(COUNTERINDUCTION) We have observed this many ravens and they all are black; so, *not* all ravens are black.

So a subjective Bayesian must concede that counterinduction is epistemically permissible as well. Similarly, by considering certain priors that equally favor Cartesian scenarios of induction and their normal counterparts, subjective

³See, for example, Carnap (1950), Hintikka (1966), and Hintikka and Niiniluoto (1980). They all talk about the probabilistic inference from evidence $F(a_1) \wedge \dots \wedge F(a_n)$ to the countable conjunction $\bigwedge_{i=1}^{\infty} F(a_i)$, where a_i means the i -th individual (or raven) observed in one's inquiry. If the index i is understood as enumerating all ravens in the universe in an order unbeknownst to the inquirer, then the evidence cannot be formalized the way they do.

⁴See, for example, Kelly (1996) and Schulte (1996).

Bayesian must also concede that it is epistemically permissible to have a degree of belief in “All ravens are black” very close to 0.5 regardless of how many black ravens have been observed in a row—namely, that it is permissible to follow the skeptical policy of “no induction”. To make those concessions explicit is to invite worries and complaints from those who would like to have a justification for full enumerative induction, *against* counterinduction, and *against* the skeptical policy. That is probably why we seldom hear subjective Bayesians talk about full enumerative induction.

Formal learning theorists fare no better. When they justify the use of an inductive principle for tackling a certain empirical problem, they justify it as a necessary means of achieving a certain epistemic ideal for tackling that problem. The epistemic ideal they have in mind pertains to finding the truth. Here is the idea. A learning method is a function from data to hypotheses. It produces a sequence of hypotheses in a possible history of inquiry—a possible way for the inquiry to unfold indefinitely. If a learning method will eventually produce the truth among the hypotheses on the table (say, “All ravens are black” and “Not all are so”) and will henceforth continue to do so in a possible history of inquiry, say that it *converges to the truth* in that possible history of inquiry (with respect to the problem of whether all ravens are problem). Convergence to the truth sounds good, although it may not be the only good thing to have. It sounds even better to have a learning method that achieves convergence to the truth in *all* possible ways for the inquiry to unfold indefinitely—that is the epistemic ideal that formal learning theorists think we should strive for, aka *identification in the limit* (Gold 1965 and Putnam 1965). When applied to the problem of whether all ravens are black, identification in the limit sets an extremely high standard: convergence to the truth in a Cartesian scenario and in its normal counterpart. But to find the truth in one of those two scenarios is to fail to do it in the other. So the epistemic standard in question—identification in the limit—is too high to be achieved by any learning method. Where it is impossible to achieve identification in the limit, nothing can be justified as a necessary means for achieving that.

So those are the reasons why normative/evaluative studies of inductive inference have said little about full enumerative induction. And we are left with the problem of giving a justification for full enumerative induction, against counterinduction, and against the skeptical policy of “no induction”. I call this problem the *Cartesian problem of induction* because of the role played by Cartesian scenarios of the sort mentioned above. The point of pursuing this problem is not to respond to every conceivable kind of inductive skeptic, but to push ourselves to the limit—to explore the extent to which we can justify full enumerative induction.

The present paper aims to take a first step toward the first positive solution to that problem. The key to my proposal is that Bayesians can and should be *learning-theoretic*, and learning theorists should have been *truly* learning-theoretic, true to what I identify as the spirit of learning theory. Here is the idea:

LEARNING-THEORETIC EPISTEMOLOGY

1. (MATHEMATICS) There are various modes of convergence to the truth for a learning method to satisfy with respect to one empirical problem or another. For example, formal learning theory studies identification in the limit, which may be called *everywhere* convergence to the truth, for it quantifies over all possible ways for the inquiry to unfold indefinitely. Statistics studies stochastic modes of convergence, such as *almost sure* convergence to the truth (aka statistical consistency).
2. (EPISTEMOLOGY) Certain modes of convergence to the truth are epistemic ideals. Some modes or their combinations are higher epistemic ideals than some others. The best learning methods for tackling an empirical problem P must have *at least* this property: achieving the highest mode of convergence to the truth that is achievable with respect to P . *Achieve the best you can have.*⁵
3. (CRUX) But for tackling the raven problem—i.e. the problem of whether all ravens are black—it is provably impossible to achieve everywhere convergence to the truth. So, to achieve the best we can have, we have to first *look for what can be achieved*. How? Well, given that it is impossible to converge everywhere, let's try to see whether it is at least possible to converge “almost everywhere”—to converge in “almost all” possible ways for the inquiry to unfold indefinitely. And it should not be difficult to try, because we do not need to do everything from scratch: mathematicians have worked out some conceptions of “almost everywhere” and “almost all”.
4. (PROPOSAL) Let's try to define various modes of convergence to the truth, including “almost everywhere” convergence, and even some joint modes, such as “almost everywhere” convergence plus “monotonic” convergence. Find the combinations that are achievable with respect to the raven problem. From among the achievable ones, identify the highest

⁵For precursors to this epistemological idea, see Reichenbach (1938), Kelly (1996), and Schulte (1996).

epistemic ideal. Then see whether a necessary condition for a learning method to achieve *that highest one* is to implement full enumerative induction rather than counterinduction or the skeptical policy. If the answer is positive, then we have a justification for that kind of induction.

5. (RESULT) Yes, the answer is positive, according to corollary 6.6. And the story just told can be retold for Bayesians.

All this is done by holding on to the following guidelines of learning theory:

Look for what can be achieved.
Achieve the best we can have.

Think and acting in the way just described is what I mean by being truly learning-theoretic. If I am right, that was how Gold (1965) and Putnam (1965) created formal learning theory, one of the earliest branches of learning theory.⁶ That is also how I am going to address the Cartesian problem of induction.

The philosophical view articulated above is what I call *learning-theoretic epistemology*, which has to be defended at greater length in a future work.⁷ This paper is devoted to developing its logical and mathematical foundation, without which the above presentation would be mere hand-waving—a sin in formal epistemology. That said, I will make several philosophical points to motivate the mathematical steps to be taken in this paper. I will also explain in detail how the mathematical results are meant to solve the Cartesian problem of induction, *assuming* learning-theoretic epistemology.

Here is the roadmap for this paper: Section 2 implements the above proposal and provides a pictorial but very informative sketch of the main results, together with their intended philosophical applications. So those who do not care about the mathematical details may read just up to the end of that section (i.e. page 14). Sections 3-7 develop the technical details. Then section 8 returns to the big picture, explaining how the idea of modes of convergence to the truth *unifies* many branches of learning theory together with learning-theoretic epistemology. The presentation of the main results is kept to a minimum. So

⁶Here is a reconstructed history. Gold and Putnam address certain mathematical or empirical problems that are impossible to solve with an effective procedure—or, in the terminology of this paper, impossible to solve by achieving *everywhere* convergence to the truth with *perfect monotonicity*. What they do is in effect to drop perfectly monotonic convergence and see whether it is at least possible to achieve everywhere convergence.

⁷But, for replies to some standard worries, see Kelly (2001, 2004) and Kelly and Glymour (2004), in which they respond to Keynes' worry that in the long run we are all dead, and Carnap's worry that no convergence criterion can constrain short-run inferential practices.

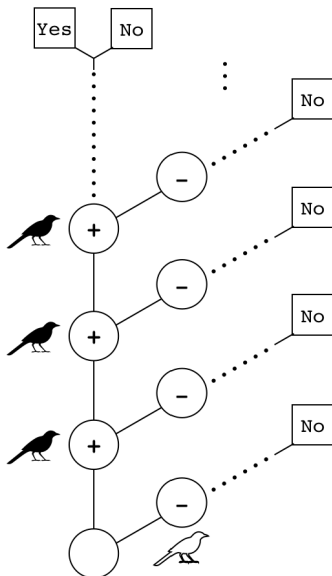


Figure 1: A partial representation of the hard raven problem

if you are interested in some of the details not required for understanding the statements of the main results (such as proofs, examples, and open problems), you will be directed to the relevant parts of the very long appendix.

To declare the style in use: Emphasis is indicated by *italics*, while the terms to be defined are presented in **boldface**.

2 Pictorial Sketch of Main Results

Here is the setting. Imagine that we are wondering whether all ravens are black. There are two competing hypotheses: **Yes** and **No**. So we are confronted with an empirical problem. Call it the **hard raven problem**. It can be partially represented by the tree in figure 1. The observation of a black raven is represented by a datum **+**; a nonblack raven, **-**. Observations of nonravens will be considered in subsequent sections but are omitted from the picture for simplicity. A data sequence (such as $(+, +, -)$) is a finite initial segment of a branch of the tree. A state of the world—i.e. a possible way for the inquiry to unfold indefinitely—is represented by an entire branch. A branch does two things: it produces an infinite data stream, such as $(+, +, +, \dots)$, and it makes one of the competing hypotheses true, i.e. either **Yes** or **No**, as indicated in the square at the tip. The Cartesian scenarios of induction discussed in

the previous section are represented by the vertical branch that makes No true. (There are actually an infinity of Cartesian scenarios if observations of nonravens are considered, such as the case in which there exists a nonblack raven but we will never see one because we will a black raven, then a white shoe, then a black raven, then a white shoe, etc.)

In general, an empirical **problem** specifies a set of competing hypotheses and a set of finite data sequences (possibly with some presuppositions that come with the problem). A **learning method** for that problem is a mapping that sends each of the finite data sequences to one of the competing hypotheses (or to the question mark that represents suspension of judgment). A learning method is evaluated in terms of its truth-finding performance in each state contained in a **state space** \mathcal{S} , which consists of all possible ways for the inquiry to unfold indefinitely (without violating the presuppositions of the problem under discussion). Each of those states makes exactly one of the competing hypotheses true and produces an infinite data stream (e_1, e_2, e_3, \dots) , to be processed incrementally by a learning method M to output a sequence of conjectures $M(e_1), M(e_1, e_2), M(e_1, e_2, e_3), \dots$ in time.

A learning method M is said to **converge to the truth** in a state $s \in \mathcal{S}$ if, in state s , method M will eventually output the true hypothesis and then always continue to do so. To achieve **everywhere** convergence to the truth is to achieve convergence to the truth in *all* states in state space \mathcal{S} . This convergence criterion is what formal learning theorists call *identification in the limit*.

The above is only one of the many convergence criteria that concern the question of *where* convergence happens, and we may also consider the question *how* convergence happens. Have a look at figure 2, in which various modes of convergence to the truth are arranged by two dimensions. The dimension that stretches to the upper right concerns where. The other dimension, which stretches to the upper left, concerns how. I introduce three modes for each of the two dimensions, so in combination there are nine modes to be considered in this paper. (Stochastic modes of convergence will not be considered because they are irrelevant to full enumerative induction.) Now I turn to explaining those modes of convergence.

A learning method M for a problem is said to achieve **almost everywhere** convergence to the truth if, for every competing hypothesis h considered in the problem, M converges to the truth in “almost all” states that make h true—or speaking geometrically, M converges to the truth “almost everywhere” on the space of the states that make h true. “**Almost everywhere**” is defined in a quite standard way in geometry and topology; it means “everywhere except on a region that is topologically negligible, i.e. nowhere dense.” A

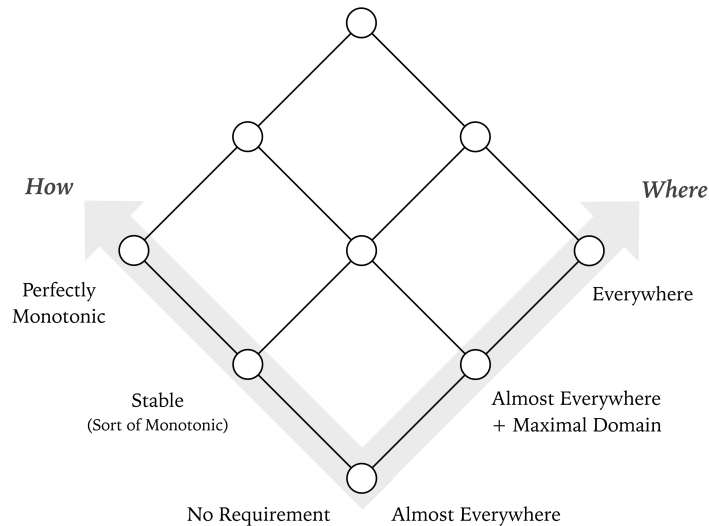


Figure 2: Modes of convergence to the truth, arranged by two dimensions

more satisfactory formal definition will be provided shortly. But, intuitively, we can think of a topologically negligible region as a slice of “hyper” Swiss cheese, which is incredibly full of holes—it is a region in the ambient space that can be constructed by removing an open set within *every* open neighborhood of *every* point in the space.

But what counts as an open neighborhood of a state in an ambient state space? I adopt what may be called **empirical topology**,⁸ which can be defined by a very natural closeness relation between states:

- (i) Consider an arbitrary state s (as depicted on the lefthand side of figure 3). An alternative state is “closer” to s iff more data are needed to distinguish it from s . The harder it is to distinguish two states empirically, the closer they are empirically.
- (ii) An open neighborhood of a state s is the set of states that are “close” to s to at least a certain degree (as depicted on the righthand side of figure 3).

Then we can prove an interesting result—lemma 4.3—which says this: Within the topological space of the states that make it false that all ravens are black, the set of the Cartesian scenarios is one of the (many) topologically

⁸Empirical topology is proposed by Vickers (1989) in computer science and by Kelly (1996) in epistemology.

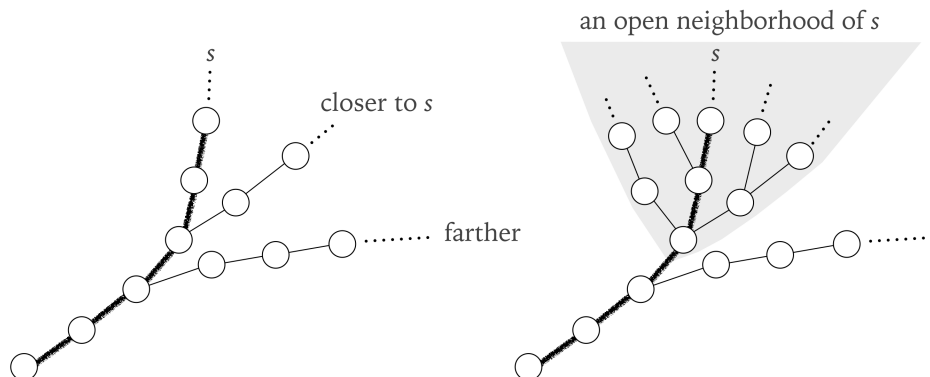


Figure 3: The empirical topology on a space of data streams or states

negligible regions. This is a lemma for proving a theorem of this paper. But I do *not* wish to infer from “*that* is topologically negligible” to “we should ignore *that*”. To infer that way is to ignore all possible states of the world, because every state forms a singleton that is topologically negligible. And it is absurd to ignore all states. A good justification for full enumerative induction has to be formulated in a more careful way.

The point just made is important, so let me reiterate it in another way. An immediate consequence of the lemma just reported is that, for any learning method tackling the hard raven problem, if it implements full enumerative induction and never applies counterinduction (which will be defined rigorously in a subsequent section), then it fails to converge to the truth *only* in Cartesian scenarios—and hence it achieves almost everywhere convergence to the truth with respect to the hard raven problem. This result sounds good but does not really suffice to justify full enumerative induction. For (as we will see in section 5) there are other learning methods that also achieve almost everywhere convergence to the truth—some are quite crazy, such as the learning method that persistently implements counterinduction and switches to full enumerative induction only after observing a white shoe or some other nonraven. So the “almost everywhere” mode of convergence to the truth, alone, is too weak to provide a justification for full enumerative induction and against counterinduction. Additional modes of convergence are needed to impose a stronger constraint on good learning methods.

Accordingly, say that a learning method converges to the truth on a **maximal domain** if there exists no learning method that converges to the truth in the same states and in strictly more states.

A learning method is said to achieve **perfectly monotonic** convergence

to the truth if, whenever it outputs one of the competing hypotheses (rather than suspending judgment), it outputs the truth and will continue to have the same output regardless of any further data received—it basically halts, just like an effective problem-solving method as studied in computability theory.

A learning method is said to achieve **stable** convergence to the truth if, whenever it outputs a *true* competing hypothesis, it will continue to have the same output.⁹ What if it outputs a falsehood? Stable convergence is silent about this case. So, to be stable is to be “sort of” monotonic but not necessarily perfectly so. Plato would probably love stable convergence to the truth, for it basically requires that, whenever the inquirer forms a belief in the true competing hypothesis, this belief is not merely a true opinion but has been “stabilized” or “tethered” to the truth, attaining the epistemic status that Plato values in *Meno*.

Finally, by ‘no requirement’ in figure 2, I mean no requirement on how to converge.

The above finishes the sketch of the three modes of convergence on each of the two axes in figure 2. So there are nine “combined” modes of convergence arranged into a two-dimensional lattice structure, in which some modes are ordered higher than some others. A mode, if ordered higher, is mathematically stronger; it implies all the modes ordered lower in the lattice. That is mathematics. The following is epistemology. I make the evaluative assumption that:

A mode of convergence to the truth, if ordered higher in the lattice in figure 2, is a higher epistemic ideal.

In fact, I even think that this assumption is obvious—or will become so after the definitions involved are rigorously stated and fully understood.¹⁰

⁹Stable convergence to the truth is closely related to some properties that have been studied in learning theory, such as: Putnam’s (1965) and Schulte’s (1996) “mind-change”; Kelly and Glymour’s (2004) “retraction”; Kelly, Genin, and Lin’s (2016) “cycle”. But these properties are defined only in terms of belief change without reference to truth. Stable convergence to the truth is a variant of the “no U-shaped learning” condition studied in Carlucci et al. (2005) and Carlucci et al. (2013), where U-shaped learning means the three-step process of believing in the truth, retracting it later, and then believing in the truth again.

¹⁰You might ask: When we compare two modes that are not ordered in the lattice—such that neither implies the other—how can we tell which corresponds to a higher epistemic ideal? This is a substantial issue, but in the present paper I do not need to take a position in order to justify full enumerative induction (although I do have an opinion, tending to think that the consideration about “where to converge” should be prioritized over the consideration about “how to converge”). See appendix A.5 for an interesting case study on this issue.

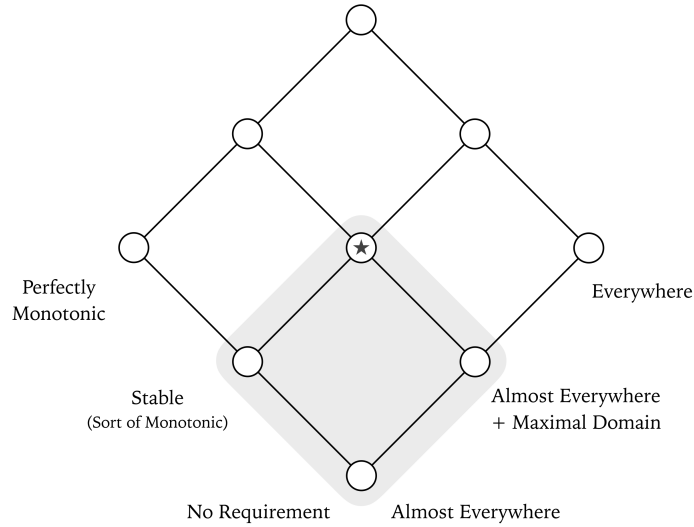


Figure 4: Modes achievable for the hard raven problem

The first main result is theorem 6.5 together with corollary 6.6, which says that, for tackling the hard raven problem, the achievable modes of convergence to the truth are the four in the shaded area of figure 4. So the highest achievable mode is the one marked by a star: “almost everywhere” + “maximal domain” + “stable”. Furthermore, the starred mode can be achieved *only* by a small range of learning methods: only those that refuse the skeptical policy, implement full enumerative induction, and never apply counterinduction. We need *all* of those three modes of convergence to the truth—“almost everywhere”, “maximal domain”, and “stable”—to narrow down the range so much.

According to learning-theoretic epistemology, the right way to justify full enumerative induction—or at least to justify its use for tackling the hard raven problem—is the following:

LEARNING-THEORETIC ARGUMENT

1. (EVALUATIVE PREMISE) In the lattice depicted in figure 4, if a mode of convergence to the truth is ordered higher, it is a higher epistemic ideal.
2. (EVALUATIVE PREMISE) For tackling an empirical problem, the best learning methods must at least achieve, of the nine epistemic ideals in the lattice, the highest achievable one—when such a highest one exists uniquely. (Achieve the best we can have!)

3. (MATHEMATICAL PREMISE) By the theorem reported above and proved in this paper:
 - 3.1 For tackling the hard raven problem, the achievable modes in that lattice are the four in the shaded area depicted in figure 4.
 - 3.2 Furthermore, the starred mode is achieved only by learning methods that implement full enumerative induction rather than counterinduction or the skeptical policy.

4. (EVALUATIVE CONCLUSION) So, the best learning methods for tackling the hard raven problem must have at least the following properties:
 - 4.1 achieving the starred mode of convergence to the truth, i.e., “almost everywhere” + “maximal” + “stable” (by 1, 2, and 3.1);
 - 4.2 implementing full enumerative induction, rather than counterinduction or the skeptical policy (by 3.2 and 4.1).

This argument has a number of features. It is a *deductively valid* argument for an *evaluative conclusion* about induction, *only* with premises that are mathematical or evaluative, *free* from any empirical premise (such as “Nature is uniform”). So, if it is possible to have *a priori* justified belief in mathematical and evaluative theses, it is tempting to wonder whether we can have *a priori* justified belief in the three premises and, hence, in the conclusion. If the answer is positive, it is also tempting to wonder whether this argument replies to any kind of inductive skeptical challenge. Those questions are very interesting and important, and must be addressed in a follow-up paper.¹¹

To clarify: The above does *not* mean that convergence to the truth is all we care about. Feel free to pursue other epistemic ideals in addition to convergence to the truth, as you see fit. This freedom is explicit in the learning-theoretic argument; just look for the keyword ‘at least’ in the second premise.

¹¹Those features of the argument—as indicated by italics in this paragraph—are key to escaping from Hume’s dilemma—or so I argue elsewhere *without* assuming learning-theoretic epistemology. The idea is to defend and develop a very general strategy due to Reichenbach (1938: sec. 39). According to the Reichenbachian strategy, a justification of induction does not have to be an argument for an *empirical* thesis about the uniformity of nature (*pace* Hume) but can be an argument for an *evaluative* thesis about how to pursue an inquiry. I call this the *evaluative turn*, and believe that it deserves more attention, although I do not think that Reichenbach’s own implementation of this general strategy is good. A better implementation is available by going Bayesian or learning-theoretic—or so I argue in unpublished manuscript Lin (2017) “Hume’s Dilemma and the Evaluative Turn”, available upon request.

The achievement of the best achievable mode of convergence to the truth is great. But it is not claimed to be sufficient for making a best learning method. It is only claimed to be necessary. Well, this *does* mean that we should take convergence to the truth very seriously. After all, if you suspect that your learning method will not give you the truth *even when* it is given an infinite amount of data, why do you want to follow it *now*?

The second main result is theorem 7.3, and one of its consequences is that, for tackling any problem, following Ockham’s razor of a certain kind is necessary for achieving any mode of convergence to the truth that is strong enough to imply “almost everywhere” + “stable”.¹² This kind of Ockham’s razor says: “Do not accept a competing hypothesis more complicated than necessary for fitting the data you have”, where a competing hypothesis is simpler (less complicated) if it is more parsimonious in terms of the capacity to fit data. In light of this result, a learning-theoretic epistemologist can argue for the following evaluative thesis:

The best learning methods for tackling a problem P must implement the kind of Ockham’s razor just mentioned if, with respect to P , the highest achievable mode of convergence to the truth implies “almost everywhere” + “stable”.

The connection between the two main results is that, as we will see, any instance of counterinductive inference violates that kind of Ockham’s razor. In fact, the second main result serves as a lemma for proving the most difficult part of the first main result.

The results sketched above all have Bayesian versions, stated and proved in this paper. Some Bayesians would not care, but some other Bayesians would and should, because there is a version of Bayesian epistemology that is learning-theoretic in nature. To be more precise, consider cognitively idealized agents—those who have sharp, real-valued degrees of belief in propositions closed under certain logical connectives (such as ‘and’, ‘or’, ‘not’, and possibly more). According to Bayesianism, those agents can be epistemically evaluated as coherent or not. Versions of Bayesianism may differ in terms of what counts as coherent—that is quite familiar. What is less familiar is that there is a version of Bayesianism that adds the following:

¹²This theorem, 7.3, extends and strengthens some aspects of my earlier work on Ockham’s razor with co-authors (Kelly, Genin, and Lin 2016) in order to cover the hard raven problem and the like. But this theorem also simplifies and weakens some other aspects in order to highlight the core idea; in particular, the concept of Ockham’s razor used here is simpler and weaker (but strong enough for present purposes).

LEARNING-THEORETIC BAYESIANISM

1. Cognitively idealized agents can also be evaluated as good or bad agents for tackling one empirical problem or another.
2. The best cognitively idealized agents for tackling an empirical problem P must have at least the following properties:
 - (i) having coherent degrees of belief and a coherent plan for updating them (presumably, the plan to conditionalize),
 - (ii) achieving the highest achievable mode of convergence to the truth (presumably, via conditionalization) with respect to P —if such a mode exists uniquely.

I call this view *learning-theoretic Bayesianism*. Clause (i) is a distinctively Bayesian thesis. Clause (ii) is a learning-theoretic thesis; in fact, it is the Bayesian version of the second premise in the learning-theoretic argument. The epistemological idea is still learning-theoretic; it is just that, now, the doxastic modeling is quantitative rather than qualitative. Learning-theoretic Bayesianism deserves a more precise formulation than provided above—together with a careful defense. But that has to be reserved for a follow-up paper.¹³

The above summarizes the main mathematical results and the way they are used to solve the Cartesian problem of induction, *assuming* learning-theoretic epistemology. The rest of this paper is devoted to the mathematical details.

3 Preliminaries: Learning Theory

This section reviews some definitions familiar in formal learning theory.

Definition 3.1. A **problem** is a triple $\mathcal{P} = (\mathcal{H}, \mathcal{E}, \mathcal{S})$ consisting of:

- a hypothesis space \mathcal{H} , which is a set of competing **hypotheses**,
- an evidence space \mathcal{E} , which is a set of finite **data sequences** (e_1, \dots, e_n) ,
- a state space \mathcal{S} , which is a set of possible **states** of the world taking this form: (h, \vec{e}) , where:
 - h , called the uniquely **true** hypothesis in this state, is an element of \mathcal{H} ;

¹³Kelly (1996: ch. 13) is a pioneer of this view.

- \vec{e} , called the **data stream** produced in this state, is an infinite sequence of data, written $\vec{e} = (e_1, e_2, e_3, \dots)$, whose finite initial segments (e_1, \dots, e_n) are all in \mathcal{E} .

The state space \mathcal{S} of a problem $(\mathcal{H}, \mathcal{E}, \mathcal{S})$ is meant to capture the *presupposition* of that problem in this way: \mathcal{S} consists of all possible ways for the inquiry to unfold indefinitely without violating the presupposition.

Definition 3.2. A **learning method** for a problem $(\mathcal{H}, \mathcal{E}, \mathcal{S})$ is a function:

$$M : \mathcal{E} \rightarrow \mathcal{H} \cup \{?\},$$

where ? represents suspension of judgment. Given each data sequence $(e_1, \dots, e_n) \in \mathcal{E}$, the output of M is written as $M(e_1, \dots, e_n)$.

Example 3.3. The **hard raven problem** poses this question: “*Are all ravens black?*” This problem is partially represented by the tree structure in figure 1 and is formally defined as follows.

- The hypothesis space \mathcal{H} is $\{\text{Yes}, \text{No}\}$, where:
 - **Yes** means that all ravens are black,
 - **No** means that not all ravens are black.
- The evidence space \mathcal{E} consists of all finite sequences of +, 0, and/or -, where:
 - datum + denotes the observation of a black raven;
 - datum -, a nonblack raven;
 - datum 0, a nonraven.
- The state space \mathcal{S} consists of all states in one of the following three categories:¹⁴
 - (a) the states (Yes, \vec{e}) in which \vec{e} is an infinite +/0 sequence (namely, an infinite sequence containing only + and 0 observations).
 - (b) the states (No, \vec{e}) in which \vec{e} is an infinite +/0 sequence.
 - (c) the states (No, \vec{e}) in which \vec{e} is an infinite +/0/- sequence that contains at least one occurrence of -.

¹⁴Well, there is a fourth category: the states (Yes, \vec{e}) in which \vec{e} contains some occurrence of - (a nonblack raven). But such states are logically impossible, so they need not be considered.

The second category (b) contains the states in which there are nonblack ravens but the inquirer will never observe one, so they are the **Cartesian scenarios of induction**.

Example 3.4. The **easy raven problem** is basically the same as the hard raven problem except that its state space consists only of the states in categories (a) and (c), ruling out the Cartesian scenarios of induction. So the easy raven problem *presupposes* that the inquirer is not living in a Cartesian scenario of induction. It poses this question: “*Suppose that you are not living in a Cartesian scenario of induction, then are all ravens black?*”

Definition 3.5. Let M be a method for a problem $\mathcal{P} = (\mathcal{H}, \mathcal{E}, \mathcal{S})$. M is said to **converge to the truth** in a state $(h, \vec{e}) \in \mathcal{S}$ if

$$\lim_{n \rightarrow \infty} M(e_1, \dots, e_n) = h,$$

namely, there exists a positive integer k such that, for each $n \geq k$, we have that $M(e_1, \dots, e_n) = h$. M is said to converge to the truth **everywhere** for $\mathcal{P} = (\mathcal{H}, \mathcal{E}, \mathcal{S})$ if it converges to the truth in every state contained in \mathcal{S} .

Then we have the following negative result:

Proposition 3.6. *Although the easy raven problem has a learning method that converges to the truth everywhere, the hard raven problem does not.*

Proof. The first part is a classic, well-known result in learning theory. To prove the second part, let \vec{e} be an infinite +/0 sequence. Consider state $s = (\mathbf{Yes}, \vec{e})$ and its Cartesian counterpart $s' = (\mathbf{No}, \vec{e})$. It suffices to note that, for any learning method M for the hard raven problem, M converges to the truth in s iff M fails to do so in s' (because these two states are, so to speak, empirically equivalent). So there is no learning for the hard raven problem that converges to the truth everywhere. \square

No one can escape such a negative result.¹⁵ Bayesians are no exceptions. The rest of this section is meant to make this clear. (But those who do not care about Bayesian epistemology can skip).

¹⁵The same impossibility result remains even if we resort to *partial* identification in the limit, a weakening of identification in the limit due to Osherson et al. (1986), which requires that, in each possible way for the inquiry to unfold indefinitely, the true hypothesis is output infinitely often and each false hypothesis is output at most finitely often.

Definition 3.7. Let a problem $\mathcal{P} = (\mathcal{H}, \mathcal{E}, \mathcal{S})$ be given. Subsets of state space \mathcal{S} are called **propositions**. Hypothesis $h \in \mathcal{H}$ and data sequence $(e_1, \dots, e_n) \in \mathcal{E}$ are understood to express the following propositions:

$$\begin{aligned} |h| &= \{(h', \vec{e}') \in \mathcal{S} : h' = h\}; \\ |(e_1, \dots, e_n)| &= \{(h', \vec{e}') \in \mathcal{S} : \vec{e}' \text{ extends } (e_1, \dots, e_n)\}. \end{aligned}$$

That is, $|h|$ is the set of states in \mathcal{S} that make hypothesis h true, and $|(e_1, \dots, e_n)|$ is the set of states in \mathcal{S} that produce data sequence (e_1, \dots, e_n) .¹⁶ Let $\mathcal{A}_{\mathcal{P}}$ denote the smallest σ -algebra that contains the above propositions for all $h \in \mathcal{H}$ and all $(e_1, \dots, e_n) \in \mathcal{E}$. Given a probability function \mathbb{P} defined on that algebra, I will write $\mathbb{P}(h)$ as a shorthand for $\mathbb{P}(|h|)$. Similarly, I will write $\mathbb{P}(e_1, \dots, e_n)$ and $\mathbb{P}(h | e_1, \dots, e_n)$, where the latter stands for conditional probability as defined the standard way.¹⁷

Definition 3.8. A **probabilistic prior** for a problem $\mathcal{P} = (\mathcal{H}, \mathcal{E}, \mathcal{S})$ is a probability function \mathbb{P} defined on σ -algebra $\mathcal{A}_{\mathcal{P}}$ with $\mathbb{P}(e_1, \dots, e_n) > 0$ for each data stream $(e_1, \dots, e_n) \in \mathcal{E}$. \mathbb{P} is said to (have its posteriors) **converge to the truth** in a state $s = (h, \vec{e}) \in \mathcal{S}$ if

$$\lim_{n \rightarrow \infty} \mathbb{P}(h | e_1, \dots, e_n) = 1,$$

that is, for any $\epsilon > 0$, there exists a positive integer k such that, for each $n \geq k$, $\mathbb{P}(h | e_1, \dots, e_n) > 1 - \epsilon$. \mathbb{P} is said to converge to the truth **everywhere** for problem $(\mathcal{H}, \mathcal{E}, \mathcal{S})$ if it converges to the truth in each state in \mathcal{S} .

Proposition 3.9. *The hard raven problem has no probabilistic prior that converges to the truth everywhere.*

Proof. Copy the proof of proposition 3.6, paste it here, replace ‘learning method’ by ‘probabilistic prior’, and replace ‘ M ’ by ‘ \mathbb{P} ’. \square

4 Preliminaries: Topology

When convergence to the truth cannot be achieved everywhere, let’s see whether it can be achieved at least *almost* everywhere. This section provides an explanation of the concept of “almost everywhere”.

¹⁶If you like, the concept of problems and other learning-theoretic concepts can be defined purely in terms of propositions, as done in Baltag et al. (2015) and Kelly et al. (2016).

¹⁷Namely, $\mathbb{P}(A | B) = \mathbb{P}(A \cap B) / \mathbb{P}(B)$.

Let a space X of data streams be given. Choose an arbitrary data stream \vec{e} therein, and take it as the actual data stream to be received incrementally by the inquirer, as depicted on the lefthand side of figure 3. Consider an alternative data stream \vec{e}' that is **identical** to \vec{e} **up until** stage n ; namely, $e'_i = e_i$ for each $i \leq n$ but $e'_{n+1} \neq e_{n+1}$. The larger n is, the later the point of departure is and the more data one needs to distinguish those two data streams. So, the larger n is, the harder it is to empirically distinguish those two data streams, and the “closer” \vec{e}' is to the actual data stream \vec{e} —“closer” in an empirical sense. Consider the set of the data streams that are at least “that close” to \vec{e} :

$$N_n(\vec{e}) = \{\vec{e}' \in X : e'_i = e_i \text{ for each } i \leq n\}.$$

Take that as a *basic open neighborhood* of point \vec{e} in space X , as depicted on the righthand side of figure 3. Such open neighborhoods provably form a *topological base* of X .¹⁸

Similarly, given the space $|h|$ of states that make a certain hypothesis h true, two states in $|h|$ are close (hard to distinguish empirically) iff the data streams therein are close. So a basic open neighborhood of a state $s = (h, \vec{e})$ in $|h|$ takes the following form:

$$\begin{aligned} N_n((h, \vec{e})) &= \{(h, \vec{e}') \in |h| : e'_i = e_i \text{ for each } i \leq n\} \\ &= |h| \cap |(e_1, \dots, e_n)|. \end{aligned}$$

Such neighborhoods provably form a topological base of $|h|$, turning $|h|$ into a topological space. This is the topological base we will use. It is determined by the empirical distinguishability between states—distinguishability in terms of (finite) data sequences.

Definition 4.1. Given a problem $\mathcal{P} = (\mathcal{H}, \mathcal{E}, \mathcal{S})$ and a hypothesis $h \in \mathcal{H}$, the **empirical topological base** of $|h|$ is the family of open neighborhoods constructed above; namely, it is defined as follows:

$$\begin{aligned} \mathcal{B}_{|h|} &= \left\{ N_n(s) : s \in |h| \text{ and } n \in \mathbb{N}^+ \right\} \\ &= \left\{ |h| \cap |(e_1, \dots, e_n)| : (e_1, \dots, e_n) \in \mathcal{E} \right\} \setminus \left\{ \emptyset \right\}. \end{aligned}$$

¹⁸Here is the standard definition of topological bases. Given a set X of points, a family \mathcal{B} of subsets of X is called a **topological base** if (i) every point in X is contained in some set in \mathcal{B} and (ii) for any $B_1, B_2 \in \mathcal{B}$ and any point $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

I now turn to some concepts borrowed from general topology.

Definition 4.2. Let X be a topological space equipped with a topological base \mathcal{B}_X . A **negligible** (or **nowhere dense**) region within X is a subset R of X such that, for each nonempty open neighborhood $N \in \mathcal{B}_X$, there exists a nonempty open neighborhood $N' \in \mathcal{B}_X$ that is nested within N and disjoint from R .

A negligible region is like a slice of “hyper” Swiss cheese, incredibly full of holes: wherever you are in the ambient space X , say point x , and however small a basic open neighborhood of x is considered, say N , then within N you can always find an open “hole” N' of that slice of Swiss cheese. Here is an example:

Lemma 4.3. *In the hard raven problem, the Cartesian scenarios of induction (i.e. the states (\mathbf{No}, \vec{e}) with \vec{e} being a $+/0$ sequence) form a negligible region within the topological space $|\mathbf{No}|$.*

Proof. Each nonempty basic open neighborhood in the topological space $|\mathbf{No}|$, say $N = |\mathbf{No}| \cap |(e_1, \dots, e_n)|$, includes a nonempty basic open neighborhood, namely $N' = |\mathbf{No}| \cap |(e_1, \dots, e_n, -)|$, which is disjoint from the set of the Cartesian scenarios of induction. \square

With “negligible”, we can define “almost everywhere” and “almost all” the standard way in topology:

Definition 4.4. Consider a property of points in a topological space X . That property is said to apply **almost everywhere** on space X if it applies to all points in $X \setminus X'$, where X' is some negligible region within X . In that case, also say that it applies to **almost all** points in space X .

See appendix A.1 for a more detailed review of “almost everywhere” in general topology.¹⁹

5 Mode (I): “Almost Everywhere”

The preceding section is purely explicatory: it provides an explication of the concept of “almost everywhere”, and gives no epistemology at all. Epistemology starts from here: using an explication of “almost everywhere” to define an epistemic ideal and to explore its significance:

¹⁹Also see Oxtoby (1996) for an elegant presentation.

Definition 5.1. A learning method M for a problem $\mathcal{P} = (\mathcal{H}, \mathcal{E}, \mathcal{S})$ is said to converge to the truth **almost everywhere** if, for each hypothesis $h \in \mathcal{H}$, M converges to the truth in almost all states that make h true—or speaking geometrically, M converges to the truth almost everywhere on topological space $|h|$.

Almost everywhere convergence can be defined in another way. Instead of requiring convergence to happen almost everywhere on each hypothesis $|h| \subseteq \mathcal{S}$, we can require convergence to happen almost everywhere on the entire state space \mathcal{S} . But we should not choose between those two definitions. To be truly learning-theoretic, we should view them as two distinct epistemic ideals for us to strive for where possible. When we can achieve both at the same time, we should do it.²⁰ It turns out that, if we were to consider both versions of almost everywhere convergence, the main theorems of this paper would remain the same.

The above definition makes possible a series of positive results. Here is the first one:

Proposition 5.2. *The hard raven problem has a learning method that converges to the truth almost everywhere.*

Proof. By the preceding result, lemma 4.3, the topological space $|\text{No}|$ has the following negligible region:

$$C = \{(\text{No}, \vec{e}) : \vec{e} \text{ is a } +/0 \text{ sequence}\},$$

which consists of the Cartesian scenarios of induction. So it suffices to an example of a learning method that converges to the truth in:

- every state in $|\text{Yes}|$,
- every state in $|\text{No}| \setminus C$.

The following learning method does the job:

M^* : “Output hypothesis **No** if you have observed a nonblack raven (-); otherwise output **Yes**.”

This finishes the proof. □

²⁰I am indebted to Alan Hájek for discussion of this idea.

Despite the above positive result, there are problems for which it is impossible to achieve almost everywhere convergence to the truth. Examples are provided in appendix A.2; their philosophical implications are briefly discussed in appendix A.3. The above positive result can be reproduced for Bayesians as follows.

Definition 5.3. A probabilistic prior \mathbb{P} is said to converge to the truth **almost everywhere** for a problem $\mathcal{P} = (\mathcal{H}, \mathcal{E}, \mathcal{S})$ if, for each hypothesis $h \in \mathcal{H}$, \mathbb{P} converges to the truth almost everywhere on topological space $|h|$.

Proposition 5.4. *The hard raven problem has a probabilistic prior that converges to the truth almost everywhere.*

Proof. Immediate from theorem 6.8, to be presented and proved below. \square

Well, the above is only a first step toward justifying full enumerative induction. For there remains a subproblem, which may be called the *problem of counterinduction*: Almost everywhere convergence to the truth, alone, is so liberal that it is witnessed also by some crazy learning methods that apply counterinduction. Here is an example:

M^\dagger : “Output hypothesis **No** if you have observed a nonblack raven (-) or everything you have observed is a black raven (+); output **Yes** if every raven you have observed is black and you have observed a nonraven (0).”

This method converges to the truth almost everywhere for the hard raven problem because it converges to the truth in:

- every state in $|\mathbf{Yes}| \setminus \{s\}$,
where $s = (\mathbf{Yes}, \text{the constant sequence of } +)$,
- every state in $|\mathbf{No}| \setminus (C \setminus \{s'\})$,
where $s' = (\mathbf{No}, \text{the constant sequence of } +)$.

$\{s\}$ is a negligible region within $|\mathbf{Yes}|$; $C \setminus \{s'\}$, within $|\mathbf{No}|$.

The learning method M^\dagger defined above applies counterinduction occasionally. It will be ruled out by the modes of convergence to be introduced below.

6 Modes (II) and (III): “Stable” and “Maximal”

Before one’s opinion converges to the truth, it might be false, or it might be true but to be retracted as data accumulate. But when one’s opinion has converged to the truth, it is “tied” to the truth and will not “run away”, which seems epistemically valuable. Plato expresses the same idea in *Meno*:

True opinions are a fine thing and do all sorts of good so long as they stay in their place, but they will not stay long. They run away from a man’s mind; so they are not worth much until you *tether* them by working out a reason. ... Once they are *tied down*, they become knowledge, and are *stable*. That is why knowledge is something more valuable than right opinion. What distinguishes the one from the other is the *tether*. (Emphasis mine.)

Hence the following definition:

Definition 6.1. A learning method M is said to **have converged** to the truth given the n -th stage of inquiry in a state $s = (h, \vec{e})$ if

$$M(e_1, \dots, e_k) = h \quad \text{for each } k \geq n.$$

With the above concept we can define the following two epistemic ideals:

Definition 6.2. A learning method M for a problem $\mathcal{P} = (\mathcal{H}, \mathcal{E}, \mathcal{S})$ is said to converge to the truth with **perfect monotonicity** if

in any state $s = (h, \vec{e})$, given any stage n such that $M(e_1, \dots, e_n) \neq ?$, M has converged to the truth.

Say that M converges to the truth with **stability** if the following (weaker) condition holds:

in any state $s = (h, \vec{e})$, given any stage n such that $M(e_1, \dots, e_n) = h$ (i.e. the truth in s), M has converged to the truth.

Stable convergence is sort of monotonic but not necessarily perfectly so, while perfect monotonicity can be very demanding. Indeed, when a learning method for a problem achieves *everywhere* convergence to the truth with *perfect monotonicity*, it is basically what a computability theorist would call an *effective procedure* for solving that problem.²¹ That combination of modes,

²¹Well, ignoring whether the learning method in question is a computable function.

“everywhere” + “perfectly monotonic”, is a great thing to have where achievable. But it is too high to be achievable for any problem that is inductive in nature, such as the hard raven problem. In fact, we have a stronger negative result:

Proposition 6.3. *For the hard raven problem, it is impossible to simultaneously achieve the following two modes of convergence to the truth: “almost everywhere” and “perfectly monotonic”.*

And the following is the last mode of convergence needed to state the first main result:

Definition 6.4. A learning method M for a problem is said to converge to the truth on a **maximal domain** if there is no learning method for the same problem that converges to the truth in all states where M does and in strictly more states.²²

Then we have the first main result:

Theorem 6.5. *The hard raven problem has a learning method that converges to the truth (i) almost everywhere, (ii) on a maximal domain, and (iii) with stability. Every such learning method M has the following properties:*

1. M is **never counterinductive** in that, for any data sequence (e_1, \dots, e_n) that has not witnessed a nonblack raven, $M(e_1, \dots, e_n) \neq \text{No}$;
2. M is **enumeratively inductive** in that, for any data stream \vec{e} that never witnesses a nonblack raven, $M(e_1, \dots, e_n)$ converges to **Yes** as $n \rightarrow \infty$.

The idea that underlies the proof of the above theorem is explained in appendix B.1, which I have tried to make as instructive as possible. You do not want to miss it if you are interested in how exactly stable convergence helps to argue against counterinduction. The proof itself is in appendix B.3.

Corollary 6.6. *Consider the modes of convergence to the truth arranged in the lattice in figure 4. The four modes in the shaded area are exactly those achievable for the hard raven problem. For a learning method to achieve the strongest of those four, namely “almost everywhere” + “maximal” + “stable”, a necessary condition is to implement full enumerative induction, rather than counterinduction or the skeptical policy.*

²²I am indebted to Konstantin Genin for bringing this concept to my attention.

This corollary follows immediately from preceding results: propositions 3.6 and 6.3 and theorem 6.5.

The rest of this section retells the essential part of the above story in Bayesian terms.

Definition 6.7. Let \mathbb{P} be a probabilistic prior for a problem $\mathcal{P} = (\mathcal{H}, \mathcal{E}, \mathcal{S})$. \mathbb{P} is said to **have started to stably converge** to the truth given stage n in state $s = (h, \vec{e}) \in \mathcal{S}$ if

1. $\mathbb{P}(h | e_1, \dots, e_n, \dots, e_{n+i})$ is monotonically increasing as a function of i defined on \mathbb{N} ,
2. $\mathbb{P}(h | e_1, \dots, e_n, \dots, e_{n+i})$ converges to 1 as $i \rightarrow \infty$.

\mathbb{P} is said to converge to the truth with **stability** if, for each hypothesis $h \in \mathcal{H}$, for each state $s = (h, \vec{e}) \in \mathcal{S}$ that makes h true, and for each stage n as a positive integer, if $\mathbb{P}(h | e_1, \dots, e_n) > 1/2$, then \mathbb{P} has started to converge to the truth given stage n in state s .

Theorem 6.8. *The hard raven problem has a probabilistic prior that converges to the truth (i) almost everywhere, (ii) on a maximal domain, and (iii) with stability. Every such probabilistic prior \mathbb{P} has the following properties:*

1. \mathbb{P} is **never counterinductive** in that, for any data sequence (e_1, \dots, e_n) that has not witnessed a nonblack raven, $\mathbb{P}(\text{No} | e_1, \dots, e_n) \leq 1/2$;
2. \mathbb{P} is **enumeratively inductive** in that, for any data stream \vec{e} that never witnesses a nonblack raven, $\mathbb{P}(\text{Yes} | e_1, \dots, e_n)$ converges to 1 as $n \rightarrow \infty$.

All the results of this section are proved in appendix B.3. The above finishes the mathematical result employed to justify full enumerative induction. I now turn to an immediate application: justification of Ockham’s razor.

7 “Almost Everywhere” + “Stable” \implies “Ockham”

I have presented a result (corollary 6.6) that can be used to justify a norm that says when to follow this methodological principle:

Do not accept a counterinductive hypothesis.

(When? At least when tackling the hard raven problem.) This section presents a similar result, which can be used to justify a norm that says when to follow this methodological principle:

Do not accept a hypothesis if it is more complicated than necessary for fitting data.

This is Ockham’s razor of a certain kind, where a hypothesis is simpler iff it is parsimonious in terms of the capacity to fit data.

To be more precise:

Definition 7.1. Let a problem $\mathcal{P} = (\mathcal{H}, \mathcal{E}, \mathcal{S})$ be given. A data sequence (e_1, \dots, e_n) and a hypothesis h therein are said to be **compatible** if the propositions they express have a nonempty overlap, which also means that there exists a state in \mathcal{S} that makes hypothesis h true and produces data sequence (e_1, \dots, e_n) . For each hypothesis $h \in \mathcal{H}$, let $\mathcal{E}(h)$ denote the set of data sequences in \mathcal{E} that are compatible with h (so $\mathcal{E}(h)$ captures the data-fitting capacity of h). The **empirical simplicity order**, written \prec , is defined on \mathcal{H} as follows: for all hypotheses h and $h' \in \mathcal{H}$,

$$h \prec h' \quad \text{iff} \quad \mathcal{E}(h) \subset \mathcal{E}(h').$$

Or in words, h is **simpler** than h' iff h “fits” strictly less data sequences than h' does. Say that h is **no more complex** than h' if $h' \not\prec h$.

In the hard raven problem, for example, the inductive hypothesis **Yes** is simpler than the counterinductive hypothesis **No**.

Definition 7.2. A learning method M for a problem $\mathcal{P} = (\mathcal{H}, \mathcal{E}, \mathcal{S})$ is said to follow **Ockham’s tenacious razor** just in case, for each hypothesis $h \in \mathcal{H}$ and each data sequence $(e_1, \dots, e_n) \in \mathcal{E}$, h is the output of M given (e_1, \dots, e_n) only if

- (*Razor Condition*) h is no more complex than any hypothesis in \mathcal{H} that is compatible with (e_1, \dots, e_n) ;
- (*Tenacity Condition*) h continues to be the output of M given any data sequence in \mathcal{E} that extends (e_1, \dots, e_n) and is compatible with h .

In other words, a learning method M follows Ockham’s tenacious razor just in case, whenever M outputs a hypothesis h , h is no more complex than necessary for fitting the available data and h will continue to be the output until it is refuted by the accumulated data. In the hard raven problem, to comply with the razor condition is exactly to be never counterinductive—to never infer **No** whenever one has not observed a nonblack raven.

Then we have the second main result:

Theorem 7.3 (Ockham Stability Theorem). *Let M be a learning method for a problem. Suppose that M converges to the truth almost everywhere. Then following two conditions are equivalent:*

1. M converges to the truth with stability.
2. M follows Ockham’s tenacious razor.

I call it the *Ockham stability theorem*. I understand its epistemological significance as follows. Almost everywhere convergence to the truth is a fundamental epistemic ideal to strive for whenever it is achievable. Convergence with stability is also good epistemically, but without almost everywhere convergence, it is not clear what value there is in achieving only stability. So, almost everywhere convergence first, stable convergence second. Given almost everywhere convergence, we might want to strive (further) for stable convergence (if that is possible), and to achieve that is *exactly* to follow Ockham’s tenacious razor, as stated in the above theorem.

An immediate application of the $1 \Rightarrow 2$ side of the Ockham stability theorem is to prove, as a corollary, the “never be counterinductive” part of theorem 6.5. For, when tackling the hard raven problem, to be never counterinductive is exactly to comply with the razor condition. So the Ockham stability theorem helps to justify a local norm of Ockham’s razor: Given that an inquirer tackles the hard raven problem, she ought to always follow Ockham’s tenacious razor and, hence, never apply counterinduction.

We can use the preceding theorem to justify the use of Ockham’s razor for tackling other problems, such as curve-fitting problems. See appendix A.4 for an example.

The rest of this section retells the essential part of the above story in Bayesian terms.

Definition 7.4. A probabilistic prior \mathbb{P} for a problem $(\mathcal{H}, \mathcal{E}, \mathcal{S})$ is said to follow **Ockham’s tenacious razor** just in case, for each hypothesis $h \in \mathcal{H}$ and each data sequence $(e_1, \dots, e_n) \in \mathcal{E}$, $\mathbb{P}(h | e_1, \dots, e_n) > 1/2$ only if

- (*Razor Condition*) h is no more complex than any hypothesis in \mathcal{H} that is compatible with the given data sequence (e_1, \dots, e_n) ;
- (*Tenacity Condition*) for any data sequence $(e_1, \dots, e_n, \dots, e_{n+n'})$ in \mathcal{E} that extends (e_1, \dots, e_n) and is compatible with h , $\mathbb{P}(h | e_1, \dots, e_{n+i})$ is monotonically increasing as a function of $i \in \{0, \dots, n'\}$.

Theorem 7.5 (Ockham Stability Theorem, Bayesian Version). *Let \mathbb{P} be a probabilistic prior for a problem that converges to the truth almost everywhere.*

Then, condition 1 below implies condition 2 below (but the converse does not hold):

1. \mathbb{P} converges to the truth with stability.
2. \mathbb{P} follows Ockham's tenacious razor.

Although the converse does not hold,²³ we might be able to formulate a stronger version of tenacity or a weaker version of stability in Bayesian terms in order to restore the equivalence between conditions 1 and 2. But that will not be attempted here. For there is no loss in application to epistemology: to justify Ockham's razor, what is really needed is just the implication relation from the epistemic ideal expressed by 1 to the methodological principle expressed by 2, which shows that the latter is necessary for achieving the former. The converse does no justificatory work. Showing that Ockham's razor achieves an epistemic ideal does not suffice to argue that one *has to* follow Ockham's razor, for there might be other means that also achieves the epistemic ideal.

All the results of this section are proved in appendix B.4.

8 Modes of Convergence to the Truth

Before I conclude, let me situate the present work in a broader context. Here is a new perspective on learning theory—or perhaps an old perspective that deserves an explicit formulation and more attention.

1. There are different modes of convergence to the truth, which correspond to different epistemic, truth-directed ideals for an inquirer to achieve where possible.
2. Learning theory, as a mathematical theory, is the general theory of various modes of convergence to the truth (or the correct learning target, which can be a concept, a function, or a predictive model). So learning theory has a number of branches, each studying a certain group of modes of convergence. I can count at least five branches:
 - (a) COMPUTABILITY THEORY studies the strongest mode of convergence to the truth, which combines *everywhere* convergence and *perfectly monotonic* convergence.

²³Here is the reason why the converse does not hold: the tenacity condition—as defined in the Bayesian framework—only requires posterior probability to remain the same or go up as data accumulate, but does not require it to go up high enough to ensure convergence to 1, let alone convergence with stability.

- (b) FORMAL LEARNING THEORY in the tradition of Gold (1965) and Putnam (1965) drops perfect monotonicity and studies *everywhere* convergence to the truth.
 - (c) For statistical problems, everywhere convergence to the truth is too high to be achievable due to the chancy nature of those problems. So STATISTICAL LEARNING THEORY (together with asymptotic statistics) studies modes of stochastic convergence to the truth. One example is *uniform convergence in probability*, which is basically Valiant’s (1984) PAC (probably approximately correct) learning criterion. Another example is *almost sure* convergence. So statistical learning theory is construed broadly to include at least the PAC learning theory and part of asymptotic statistics.
 - (d) For the hard raven problem and the like, everywhere convergence to the truth is too high to be achievable due to the presence of two empirically indistinguishable states of the world. So TOPOLOGICAL LEARNING THEORY, the branch I develop here, studies *almost everywhere* convergence to the truth and possibly other modes of topological convergence.²⁴
 - (e) BAYESIAN LEARNING THEORY studies the Bayesian versions of all the above modes of convergence.²⁵ Instead of convergence of qualitative beliefs to the truth, it studies convergence of probabilistic belief to full certainty in the truth.²⁶
3. Learning-theoretic epistemology is meant to provide justifications for local norms of inductive inference—“local” in the sense of being sensitive to the problem tackled by the inquirer.²⁷ But the justifications are systematically based on (i) a system of epistemic values and (ii) a general epistemic norm. So this epistemology has two parts, one evaluative and the other normative.

²⁴For example, it is possible to talk about the almost sure convergence of a learning method with respect to almost all chance distributions, taken as points in a topological space, whose topology is generated by the total variation metric between chance distributions. That can be called *almost-everywhere almost-sure* convergence to the truth, which I conjecture to be achievable for many causal discovery problems that do not make the causal faithfulness assumption or the like. This is work in progress.

²⁵For an earlier advocate of learning theory for Bayesian inquirers, see Kelly (1996: chap. 13).

²⁶Where convergence to full certainty in the truth is too demanding, we should see whether it is possible to achieve at least convergence to high credence in the truth.

²⁷For discussion of this kind of sensitivity, see appendix A.3, which presupposes appendix A.2.

- (i) The evaluative part determines which mode of convergence to the truth (or which combination of modes) corresponds to a higher epistemic ideal for an inquirer to achieve where possible. The present paper provides an example, as depicted in the lattice in figure 2.
- (ii) The normative part operates with this general guideline: “Look for what can be achieved; achieve the best you can.” The present paper provides a case study on the hard raven problem, full enumerative induction, and Ockham’s razor.

In a nutshell, it is the idea of modes of convergence to the truth that *unifies* the many branches of learning theory, together with learning-theoretic epistemology. And it is from this unificatory perspective that I develop my solution to the Cartesian problem of induction.

9 Conclusion

For tackling the problem of whether all ravens are black, the highest achievable epistemic ideal (among the ideals considered in this paper) is the combination of three modes of convergence to the truth: “almost everywhere” + “maximal domain” + “stable”, as depicted in the lattice in figure 4. And a necessary condition for achieving that is to follow full enumerative induction rather than counterinduction or the skeptical policy.

I believe that learning-theoretic epistemology is a promising approach to normative studies of inductive inference, but I will have to defend this philosophical claim at greater length in a future work. The goal of this paper is, instead, to develop a solution to the Cartesian problem of induction, to lay its logical and mathematical foundation, to help spark the development of competing solutions for future comparison, and—last but not least—to do all this from a unificatory perspective on learning theories and learning-theoretic epistemology, unified by the idea of modes of convergence to the truth.

10 Acknowledgements

This paper would have been impossible without my mentor and friend Kevin T. Kelly. I thank him for the innumerable discussions that inspired many ideas in this paper. I am indebted to Alan Hájek and Julia Staffel for discussions on an earlier draft and for their encouragement and support during my pursuit of this project. I am indebted to Konstantin Genin, I-Sen Chen, Conor Mayo-Wilson,

Liam Bright, and Ted Shear for their very detailed and helpful comments on an earlier draft. I am also indebted to the following people for discussions: Jason Konek, Ray Briggs, Ben Levenstein, Tina Rulli, David Copp, Alexandru Baltag, and Harvey Lederman. The present project was initiated when I taught learning theory in the Foundations Seminar on Formal Methods at the Australian National University, June 2015; I thank the participants for sparking this project.

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A Examples, Discussions, and Open Questions

This section contains materials that might be of interest to some but not all readers.

A.1 Review of “Almost Everywhere” in Topology

Let X be a topological space (equipped with a distinguished topological base). Let π be a property that may or may not apply to points in X . The following conditions are equivalent:

1. π applies to almost all points in X —or speaking geometrically, π applies almost everywhere on X .
2. Every nonempty (basic) open set of X has a nonempty (basic) open subset on which π applies everywhere.
3. The set of points to which π applies is comprehensive enough to include a dense open subset of X .

The equivalence between conditions 1 and 2 is used in some of the proofs in this paper. Condition 3 emphasizes the fact that “being a dense subset of X ” alone does not suffice for “containing almost all points in X ”. For example, the set of rationals is dense in the set of reals, but the former is too small to include an open subset of the latter. So the property of being a rational does not apply almost everywhere on the real line.

Sometimes topologists adopt a more lenient criterion of “almost all”, according to which a property π is said to apply to almost all points in X just in case π applies to all points in $X \setminus X'$, where X' is a countable union of negligible (i.e. nowhere dense) subsets of X . This more lenient criterion is used for proving the well-known theorem that almost all continuous functions defined on the unit interval are nowhere differentiable.²⁸

The present paper adopts the more stringent criterion of “almost everywhere”, requiring that X' be a negligible subset of X . This choice is made

²⁸See Oxtoby (1996).

for a reason that is both epistemological and exploratory. The more stringent convergence criterion corresponds to a higher epistemic ideal. I propose to see whether the higher ideal is achievable for the hard raven problem, and the answer is positive. If that were too high to be achievable, I would try to see whether the lower ideal is achievable.

A.2 Too Hard to Achieve “Almost Everywhere”

A problem can be too hard to allow for the achievement of almost everywhere convergence to the truth. There are multiple ways of generating such problems. A Cartesian skeptic has one way to offer, making use of two empirically equivalent hypotheses:

Example A.1. The **very hard raven problem** poses the following joint question:

Are all ravens black? If not, will all the ravens observed in the future be black?

There are three potential answers: **Yes**, **NoYes**, and **NoNo**. Note that **NoYes** is a Cartesian skeptical hypothesis, a hypothesis that is akin to (but not as terrible as) the proposition that one is a brain in a vat. Hypotheses **Yes** and **NoYes** are empirically equivalent—they are compatible with exactly the same data sequences. This problem can be formally defined as follows:

- the hypothesis space \mathcal{H} is $\{\mathbf{Yes}, \mathbf{NoYes}, \mathbf{NoNo}\}$,
- the evidence space \mathcal{E} consists of all finite sequences of +, 0, and/or −,
- the state space \mathcal{S} consists of all states in the following three categories:
 - (a) the states (\mathbf{Yes}, \vec{e}) in which \vec{e} is an infinite +/0 sequence,
 - (b) the states $(\mathbf{NoYes}, \vec{e})$ in which \vec{e} is an infinite +/0 sequence.
 - (c) the states (\mathbf{NoNo}, \vec{e}) in which \vec{e} is an infinite +/0/− sequence that contains at least one occurrence of −.

Then we have this negative result:

Proposition A.2. *For the very hard raven problem, it is impossible to achieve almost everywhere convergence to the truth.*

Sketch of Proof. Suppose for *reductio* that some learning method M converges to the truth almost everywhere for the very hard raven problem. By almost everywhere convergence on the space $|\mathbf{Yes}|$, there exists a $+/0$ sequence (e_1, \dots, e_n) such that M converges to the truth everywhere on $|\mathbf{Yes}| \cap |(e_1, \dots, e_n)|$. By almost everywhere convergence on the space $|\mathbf{NoYes}|$, (e_1, \dots, e_n) can be extended to some $+/0$ sequence $(e_1, \dots, e_n, \dots, e'_n)$ such that M converges to the truth everywhere on $|\mathbf{NoYes}| \cap |(e_1, \dots, e_n, \dots, e'_n)|$. Choose an (infinite) data stream $\vec{e} \in |(e_1, \dots, e_n, \dots, e'_n)|$. So:

$$\begin{aligned} (\mathbf{Yes}, \vec{e}) &\in |\mathbf{Yes}| \cap |(e_1, \dots, e_n)|, \\ (\mathbf{NoYes}, \vec{e}) &\in |\mathbf{NoYes}| \cap |(e_1, \dots, e_n, \dots, e'_n)|. \end{aligned}$$

It follows that M converges to the truth both in state (\mathbf{Yes}, \vec{e}) and in state $(\mathbf{NoYes}, \vec{e})$. But that is impossible because those two states are empirically indistinguishable and make distinct hypotheses true. \square

Due to that negative result, learning-theoretic epistemologists make no normative recommendation as to how to tackle the very hard raven problem. This raises some philosophical worries and questions, especially about the nature and purpose of learning-theoretic epistemology—see the next subsection, A.3, for a short philosophical discussion.

Here I would like to give more examples to show that, to construct a problem for which almost everywhere convergence is unachievable, it is not necessary to invoke two empirically indistinguishable *states*, and it is not sufficient to invoke two empirically equivalent *hypotheses*.

Example A.3. The **cardinality problem** poses the following question:

Given that the incoming data stream will be a 0/1 sequence, how many occurrences of 1 will there be? Zero, one, two, . . . , or infinite?

This problem can be formally defined as follows:

- the hypothesis space \mathcal{H} is $\{0, 1, 2, \dots, \infty\}$,
- the evidence space \mathcal{E} consists of all finite sequences of 0 and/or 1,
- the state space \mathcal{S} consists of all states of the following form:
 - the states (\mathbf{n}, \vec{e}) in which \mathbf{n} is a natural number and \vec{e} is a 0/1 sequence that contains exactly \mathbf{n} occurrences of 1;
 - the states (∞, \vec{e}) in which \vec{e} is a 0/1 sequence that contains infinitely many occurrences of 1.

In the above problem, any two states are empirically distinguishable, but we still have the following negative result:

Proposition A.4. *For the cardinality problem, it is impossible to achieve almost everywhere convergence.*

Sketch of Proof. Suppose for *reductio* that there exists a learning method M that converges to the truth almost everywhere for the cardinality problem. So, in particular, M converges to the truth ∞ almost everywhere in topological space $|\infty|$. It follows that, for some finite data sequence σ_* , M converges to the truth ∞ everywhere on basic open set $|\infty| \cap |\sigma_*|$. Let \mathbf{k} be the least hypothesis compatible with σ_* . By the forcing lemma (in appendix B.2), there exists a data sequence σ_k that extends σ_* and is compatible with hypothesis \mathbf{k} such that $M(\sigma_k) = \mathbf{k}$. Continue applying the forcing lemma to obtain this result: for each $n \geq k$, data sequence σ_n is extended into data sequence σ_{n+1} compatible with hypothesis $\mathbf{n+1}$ such that $M(\sigma_{n+1}) = \mathbf{n+1}$. Let σ be the infinite data sequence that extends σ_n for all natural numbers $n \geq k$. Then it is not hard to argue that M fails to converge to the truth in state $s = (\infty, \sigma)$. But this state s is in basic open set $|\infty| \cap |\sigma_*|$. Contradiction. \square

The presence of two empirically equivalent hypotheses, alone, does not imply the impossibility of achieving almost everywhere convergence. Here is a counterexample:

Example A.5. The **even-vs-odd problem** poses the following question:

Given that the incoming data stream will be a 0/1 sequence with finitely many occurrences of 1, will there be evenly many or oddly many?

This problem can be formally defined as follows:

- the hypothesis space \mathcal{H} is $\{\mathbf{Even}, \mathbf{Odd}\}$,
- the evidence space \mathcal{E} consists of all finite 0/1 sequences,
- the state space \mathcal{S} consists of all states of the following form:
 - the states (\mathbf{Even}, \vec{e}) in which \vec{e} is a 0/1 sequence that contains evenly many occurrences of 1;
 - the states (\mathbf{Odd}, \vec{e}) in which \vec{e} is a 0/1 sequence that contains oddly many occurrence of 1.

The two competing hypotheses, **Even** and **Odd**, are empirically equivalent because no data sequence refutes one and saves the other. But we still have the following positive result:

Proposition A.6. *For the even-vs-odd problem, it is possible to achieve convergence to the truth everywhere—and, a fortiori, almost everywhere.*

Sketch of Proof. Everywhere convergence is achievable for this problem, as witnessed by this method: “Output **Even** if you have observed evenly many occurrences of 1; otherwise output **Odd**.” □

A.3 Sensitivity to the Chosen Set of Hypotheses

It was remarked earlier that, for the very hard raven problem, it is even impossible to achieve almost everywhere convergence. As a consequence, learning-theoretic epistemologists have been unable to make a normative recommendation for an inquirer tackling that problem. Let me say why they have nothing to apologize.

The very hard raven problem embodies not just the philosophical problem of responding to the inductive skeptic, but also the problem of responding to the Cartesian skeptic, as highlighted by the two empirically equivalent hypotheses put on the table:

- **Yes:** “Yes, all ravens are black.”
- **NoYes:** “No, not all ravens are black; and yes, all ravens to be observed are black.”

Learning-theoretic epistemology is not designed to respond to the Cartesian skeptic, and we may conjoin it with a good, independent reply to the Cartesian skeptic. To be sure, learning-theoretic epistemologists can, and should, insist that when an inquirer tackles the hard raven problem *rather than* the very hard one, she ought to be inductive and never be counterinductive, thanks to the argument and the results provided above.

So learning-theoretic epistemology typically makes a normative recommendation of this form: “*If* one tackles such and such a problem, one ought to follow a learning method having such and such properties.” Such a normative recommendation is *sensitive* to, or *conditional* upon, the problem pursued by the inquirer.

In fact, learning theorists have long recognized that epistemology needs such sensitivity, for a reason that is not tied to Cartesian skepticism but can be traced back to the genesis of learning theory. The development of learning

theory was historically motivated by the observation that it is mathematically impossible for us learn everything by meeting the epistemic ideal of everywhere convergence to the truth. That is, it is provably impossible to design a learning machine that is so powerful as to be capable of convergently solving the “ultimate” problem, the problem that entertains all hypotheses that human beings can understand (Putnam 1963). The cardinality problem corresponds to one such example, which makes the point even without invoking two empirically indistinguishable states or a Cartesian-like demon who is always hiding some observable items from the inquirer.

Given that it is impossible to design a learning machine for learning everything that one can understand, one has to prioritize certain things to learn. That is, in a context of inquiry, an inquirer has to identify the hypotheses whose truth values she really wants to learn, and to pursue the problem consisting of those hypotheses. When she switches to a different context of inquiry, she might need to reconsider the priority and decide to pursue a different problem. For example, an inquirer in a philosophy seminar on Cartesian skepticism might take the very hard raven problem to be of the utmost importance and decide to pursue it. But when she returns to the laboratory, the only important problem to pursue might just be the hard raven problem, rather than the very hard one. Here I only claim that she might switch that way. As to whether she ought to switch that way or is at least epistemically permitted to switch that way, the positive answer has to be defended elsewhere.

To sum up, learning-theoretic epistemologists recognize two groups of important issues to address:

(Mathematical Issues) What can be learned? Which set of hypotheses can be learned in the limit? Which problem can be solved with which combinations of modes of convergence to the truth?

(Normative Issues) One has no alternative but to prioritize certain things to learn. But which to prioritize? Which hypotheses and which problem are the things that one really cares about, or ought to care about, in which context of inquiry, such as a laboratory or a philosophy seminar?

While the mathematical issues have driven the development of learning theory, learning-theoretic epistemologists still have a lot to do to address the normative issues.

A.4 How to Justify Ockham’s Razor: One More Example

Here is one more example that illustrates the application of theorem 7.3 to justification of Ockham’s razor:

Example A.7. Let x and y be real-valued variables, and suppose that y depends functionally on x . The **hard polynomial degree problem** poses the following question:

Given that y is a polynomial function of x , what is the degree of that polynomial function?

This problem considers “rectangular” data on the x - y plane. A rectangular datum e_i is an open rectangle on the x - y plane that is axis-aligned and has only rational endpoints. Understand e_i to say: “The true polynomial function passes through rectangle e_i .” A (finite or infinite) sequence of such rectangles is said to be compatible with a polynomial function if that polynomial function passes through all rectangles therein. This problem can be formally defined as follows:

- the hypothesis space \mathcal{H} is the set of possible polynomial degrees, $\{0, 1, 2, \dots\}$;
- the evidence space \mathcal{E} is the set of finite sequences of rectangular data that are compatible with at least some polynomial function;
- the state space \mathcal{S} is the set of states taking the following form:

$$(\mathbf{d}, \vec{e}),$$

where \mathbf{d} is a polynomial degree in \mathcal{H} and \vec{e} is an infinite sequence of rectangular data that is compatible with at least one polynomial function of degree \mathbf{d} .

A hypothesis of a lower polynomial degree is simpler:

$$0 \prec 1 \prec 2 \prec \dots \prec \mathbf{n} \prec \mathbf{n}+1 \dots$$

Here is an example of a learning method that follows Ockham’s tenacious razor:

M_{ock}^* “Output degree \mathbf{d} whenever \mathbf{d} is the lowest polynomial degree that can fit the data you have (namely, whenever the data sequence you have is compatible with some polynomial function of degree \mathbf{d} but with no polynomial function of any lower degree).”

This method never suspends judgment. There are other methods that also follow Ockham’s tenacious razor, and they differ from the previous one by being less opinionated, willing to suspend judgment occasionally before jumping to a conclusion.

Before we apply theorem 7.3 to the hard polynomial degree problem, we have to figure out what can be achieved for that problem:

Proposition A.8. *For the hard polynomial degree problem, it is possible to achieve convergence to the truth almost everywhere with stability on a maximal domain.*

Sketch of Proof. The existential claim is witnessed by the method M_{ock}^* defined above, and can be proved in a way that mimics the proof of the existential claim of theorem 6.5. \square

For the hard polynomial problem, is it possible achieve a higher mode, such as one that implies everywhere convergence or perfectly monotonic convergence? The answer is negative. To secure at least almost everywhere convergence, perfectly monotonic convergence is impossible because the problem in question is essentially an inductive problem, with a hypothesis that goes beyond the logical consequences of data. Everywhere convergence is unachievable because the state space is liberal enough to allow for two empirically indistinguishable states that make distinct hypotheses true. For example, consider a data stream \vec{e} being so unspecific that it is compatible with some polynomial function of degree d and also with some other polynomial function of degree $d+1$. So there are (at least) two empirically indistinguishable states that make distinct hypotheses true, namely (d, \vec{e}) and $(d+1, \vec{e})$. Therefore, this problem does not have a learning method that converges to the truth everywhere. This is why it is called a “hard” problem, which suggests that we can obtain an “easy” version if we are willing to make a sufficiently strong presupposition to constrain the state space.²⁹

So here is what we have: When tackling the hard polynomial problem, an inquirer ought to achieve the highest achievable epistemic ideal among those in the lattice in figure 4, and that is the joint mode of convergence to the truth “almost everywhere” + “stable” + “maximal”. A necessary condition

²⁹To be more specific, the **easy polynomial degree problem** is the same as the hard one except that it has a more constrained state space, in which each state (d, \vec{e}) is required to be such that its data stream \vec{e} is compatible with exactly one polynomial function and that unique polynomial function has degree d . For this problem, everywhere convergence to the truth is achievable.

for achieving that is to follow Ockham’s tenacious razor, thanks to theorem 7.3. This is why an inquirer tackling the hard polynomial degree problem ought to follow Ockham’s tenacious razor—or so I submit.

You might wonder whether Ockham’s razor can be justified in a simpler way than I just did. Suppose that we have proved that a problem \mathcal{P} is easy enough to make it possible to achieve at least “almost everywhere” + “stable”, setting aside all other modes of convergence. Then it is tempting to quickly conclude that \mathcal{P} ought to be tackled with a method that achieves “almost everywhere” + “stable”, and then immediately apply theorem 7.3 to conclude that \mathcal{P} ought to be tackled with a method that follows Ockham’s tenacious razor. It is tempting to do all this and rush to justify Ockham’s razor, without considering higher epistemic ideals, such as one that adds convergence on a maximal domain. Is it OK to rush to justify Ockham’s razor that way?

The answer is negative, and it is important to know why:³⁰ Some problems involve a trade-off between “stable” and “maximal” that forces the inquirer to sacrifice one in order to secure the other—see appendix A.5 for an example. In that case, it may not be immediately clear as to whether the inquirer should opt for the package “almost everywhere” + “stable” or side with “almost everywhere” + “maximal”. Only the former requires following Ockham’s tenacious razor; the latter does not.

A.5 Trade-off Between “Stable” and “Maximal”

There are situations in which the inquirer is, in a sense, forced to make a trade-off between two desirable modes of convergence, such as stability and maximality. Here is an example:

Example A.9. The **bounded even-vs-odd problem** poses the following question.³¹

Given that the incoming data stream will be a 0/1 sequence with at most two occurrences of 1, will there be evenly or oddly many occurrences of 1?

Then, this problem can be formally defined as follows:

- the hypothesis space \mathcal{H} is $\{\text{Even}, \text{Odd}\}$,

³⁰The point made in this paragraph is a supplement to the way that I and co-authors justify Ockham’s razor in Kelly, Genin, and Lin (2016). In that earlier work, the achievability of “everywhere convergence” plus “cycle-free” (a variant of stability) is taken to be sufficient for justifying the use of Ockham’s razor.

³¹I thank Konstantin Genin for bringing this problem to my attention.

- the evidence space \mathcal{E} consists of all finite 0/1 sequences that have at most two occurrences of 1,
- the state space \mathcal{S} consists of all states of the following form:
 - the states (**Even**, \vec{e}) in which \vec{e} is a 0/1 sequence that contains exactly zero or two occurrences of 1;
 - the states (**Odd**, \vec{e}) in which \vec{e} is a 0/1 sequence that contains exactly one occurrence of 1.

In this problem, **Odd** is simpler than **Even**.

Then we have the following trade-off result:

Proposition A.10. *Consider the following two modes of convergence:*

- (1) *convergence to the truth with stability,*
- (2) *convergence to the truth on a maximal domain.*

For the bounded even-vs-odd problem, each of those modes is achievable, but they are not jointly achievable.

Sketch of Proof. Everywhere convergence is achievable for this problem, as witness by this method: “Output **Even** if you have observed evenly many occurrences of 1; otherwise output **Odd**.” As a consequence, convergence on a maximal domain is equivalent to everywhere convergence. Suppose that M converges to the truth on a maximal domain. So M converges to the truth everywhere and, hence, in the states (**Even**, \vec{e}) with \vec{e} containing no occurrence of 1. But convergence in those states can be argued to violate the razor condition in Ockham’s tenacious razor. Then, by the Ockham stability theorem 7.3, M fails to achieve stable convergence. \square

Fortunately, for the bounded even-vs-odd problem, the inquirer is forced to choose between stability and maximality only in a weak sense: she could have easily fine-grain the hypothesis space and ask instead: “How many occurrences of 1 will there be? Zero, one, or two?” Call this fine-grained problem the **zero-vs-one-vs-two problem**. For this problem, stability and maximality are jointly achievable together with everywhere convergence. This observation leads to the following open questions:

(*Open Questions*) Are there problems for which it is possible to achieve stability, possible to achieve maximality, impossible to achieve both simultaneously, and even impossible to achieve both no matter how the hypothesis space is fine-grained? If there are such problems, which mode of convergence should one sacrifice in exchange for the other?

I tend to think that, in such an unfortunate problem, stability should be sacrificed in exchange for maximality—in general, the consideration about “where to converge” should be prioritized over the consideration about “how to converge”. But this normative claim will have to be defended in another paper.

B Proofs

B.1 The Idea of Proof of Theorem 6.5

Theorem 6.5 has three parts. The existential claim and the universal claim about “be enumeratively inductive” are two easier parts. The crucial part is the universal claim about “never be counterinductive”. The reason why it is crucial is two-fold: first, it is a very instructive special case of the $1 \Rightarrow 2$ side of the Ockham stability theorem 7.3; second, it is a lemma for proving the part about “be enumeratively inductive”. So let me separate the crucial part for a closer examination:

Proposition B.1 (Never Be Counterinductive). *Let M be a learning method for the hard raven problem that converges to the truth almost everywhere with stability. Then M is never counterinductive.*

Note that we do not need convergence on a maximal domain here. It will be convenient to have the following concept:

Definition B.2. Given a problem $(\mathcal{H}, \mathcal{E}, \mathcal{S})$, a data sequence $(e_1, \dots, e_n) \in \mathcal{E}$ is said to be **compatible** with a hypothesis $h \in \mathcal{H}$ if the propositions they express have a nonempty overlap, namely:

$$|h| \cap |(e_1, \dots, e_n)| \neq \emptyset,$$

which also means that (e_1, \dots, e_n) can be extended into a data stream \vec{e} such that (h, \vec{e}) is a state in \mathcal{S} .

The proof of the above proposition proceeds as follows. Let M be a learning method for the hard raven problem that converges to the truth almost

everywhere. Suppose that M is sometimes counterinductive, namely, for some $+/0$ sequence (e_1, \dots, e_n) , we have that:

$$M(e_1, \dots, e_n) = \text{No}. \quad (1)$$

It suffices to show that M fails to converge to the truth with stability. Since (e_1, \dots, e_n) is a $+/0$ sequence, it is compatible with **Yes**. To summarize, we have had:

- M converges to the truth almost everywhere.
- (e_1, \dots, e_n) is compatible with **Yes**.

Given these two conditions, we can apply the so-called *forcing lemma* (to be stated soon) in order to “force” M to output **Yes** by extending (e_1, \dots, e_n) into a certain $+/0$ sequence $(e_1, \dots, e_n, \dots, e_{n'})$ such that:

$$M(e_1, \dots, e_n, \dots, e_{n'}) = \text{Yes}. \quad (2)$$

Now, choose a state s such that:

$$s \in |\text{No}| \cap |(e_1, \dots, e_n, \dots, e_{n'})|. \quad (3)$$

We can always make this choice because every data sequence is compatible with **No**. By (1)-(3), we have: given the earlier stage n in state s , M outputs the truth **No** but fails to have converged to the truth. So M does not converge to the truth with stability. This finishes the proof of the part “never be counterinductive” in theorem 6.5—as soon as the forcing lemma is stated and established.

Let me state the forcing lemma here, remark on its importance, and leave its proof to appendix B.2:

Lemma (Forcing Lemma). *Let $(\mathcal{H}, \mathcal{E}, \mathcal{S})$ be an arbitrary problem. Suppose that M is a learning method for it that converges to the truth almost everywhere, and that $(e_1, \dots, e_n) \in \mathcal{E}$ is compatible with $h \in \mathcal{H}$. Then the above data sequence can be extended into a data sequence $(e_1, \dots, e_n, \dots, e_{n'}) \in \mathcal{E}$ such that:*

1. $(e_1, \dots, e_n, \dots, e_{n'})$ is still compatible with h ,
2. $M(e_1, \dots, e_n, \dots, e_{n'}) = h$.

This lemma has a weaker and classic version, which deletes ‘almost’ and applies only to learning methods that converge to the truth everywhere. The weaker version has played an important role in proving many results in formal learning theory. Now, with the forcing lemma strengthened to cover almost everywhere convergence, many old proof techniques can be carried over to the learning theory developed here.³² In fact, most of the results of this paper—positive or negative—are proved with the help of the forcing lemma.

Now let me turn to sketching the proof of the part “be enumeratively inductive”. Suppose that learning method M converges to the truth almost everywhere with stability (and we are going to suppose that M converges on a maximal domain only when we really need to). Then, by the preceding result, M is never counterinductive, and hence it fails to converge to the truth in every Cartesian scenario of induction, say (No, \vec{e}) , where \vec{e} contains no occurrence of a nonblack raven. This failure of convergence in the Cartesian state (No, \vec{e}) opens the possibility for M to converge to the truth in its normal counterpart (Yes, \vec{e}) . To turn this possibility into a reality, it suffices to invoke the last supposition of the theorem, that M converges to the truth on a maximal domain. It can be shown that, in order for M to converge to the truth on a maximal domain, the domain of convergence of M has to be so comprehensive that it contains all states that make hypothesis **Yes** true, which implies that M is enumeratively inductive.

As to the proof of the existential claim, it is almost routine to verify that it is witnessed by the method M^* constructed above, which says: “Output hypothesis **No** if you have observed a nonblack raven (-); otherwise output **Yes**.”

This finishes the proof sketch of theorem 6.5.

B.2 Proof of the Forcing Lemma

The forcing lemma has two versions, one for (qualitative) learning methods and the other for probabilistic priors.

Lemma B.3 (Forcing Lemma, Qualitative Version). *Let $(\mathcal{H}, \mathcal{E}, \mathcal{S})$ be an arbitrary problem. Suppose that M is a learning method for it that converges to the truth almost everywhere, and that $(e_1, \dots, e_n) \in \mathcal{E}$ is compatible with $h \in \mathcal{H}$. Then the above data sequence can be extended into a data sequence $(e_1, \dots, e_n, \dots, e_{n'}) \in \mathcal{E}$ such that:*

³²In case you are interested: the forcing lemma can even be strengthened further to apply to *convergence to the truth on a dense set*. I wonder whether such a weak convergence criterion is interesting epistemologically, but I will not address this question here.

1. $(e_1, \dots, e_n, \dots, e_{n'})$ is still compatible with h ,
2. $M(e_1, \dots, e_n, \dots, e_{n'}) = h$.

Proof. Suppose that (e_1, \dots, e_n) is compatible with h . Namely,

$$|h| \cap |(e_1, \dots, e_n)|$$

is a nonempty basic open set of topological space $|h|$. We are going to make use of the following characterization of “almost everywhere” in general topology:

A property applies almost everywhere on a topological space (with a distinguished topological base) if, and only if, each nonempty (basic) open set U has a nonempty (basic) open subset U' such that the property applies everywhere on U' .

So, by the “only if” side and the hypothesis that M converges to the truth almost everywhere, it follows that $|h| \cap |(e_1, \dots, e_n)|$ has a nonempty basic open subset:

$$|h| \cap |(e_1, \dots, e_n, \dots, e_k)|$$

on which M converges to the truth everywhere. Now, within this nonempty set, choose an arbitrary state (h, \vec{e}) . So, in that state, M converges to the truth. Then there exists a positive integer $n' \geq k$ such that M outputs the truth h given the n' -th stage along data stream \vec{e} . That is:

$$M(e_1, \dots, e_n, \dots, e_k, \dots, e_{n'}) = h.$$

It is not hard to see that the input is still compatible with h . □

Lemma B.4 (Forcing Lemma, Bayesian Version). *Let $(\mathcal{H}, \mathcal{E}, \mathcal{S})$ be an arbitrary problem. Suppose that \mathbb{P} is a probabilistic prior for it that converges to the truth almost everywhere, and that $(e_1, \dots, e_n) \in \mathcal{E}$ is compatible with $h \in \mathcal{H}$. Then the above data sequence can be extended into a data sequence $(e_1, \dots, e_n, \dots, e_{n'}) \in \mathcal{E}$ such that:*

1. $(e_1, \dots, e_n, \dots, e_{n'})$ is still compatible with h ,
2. $\mathbb{P}(h | e_1, \dots, e_n, \dots, e_{n'}) > 1/2$.

Proof. Copy the proof of the qualitative version of the forcing lemma, and paste it here. Now, replace the only occurrence of $M(e_1, \dots, e_n, \dots, e_k, \dots, e_{n'}) = h$ by $\mathbb{P}(h | e_1, \dots, e_n, \dots, e_k, \dots, e_{n'}) > 1/2$. As the last step, replace each occurrence of M by \mathbb{P} . □

B.3 Proofs for Section 6: Enumerative Induction

The proofs presented in this section rely on the forcing lemma proved in section B.2.

Proof of Proposition 6.3. Suppose that a learning method M for the hard raven problem achieves perfectly monotonic convergence. Then M is a “non-inductive” method in that it never outputs **Yes**, so it fails to converge to the truth in every state in $|\mathbf{Yes}|$. So M fails to converge to the truth almost everywhere in the topological space $|\mathbf{Yes}|$. So M fails to converge to the truth almost everywhere. \square

Proof of Theorem 6.5. To establish the existential claim, it suffices to show that it is witnessed by the learning method M^* we have discussed: “Output hypothesis **No** if you have observed a nonblack raven (-); otherwise output **Yes**.” Proposition 5.2 has established that M^* converges to the truth almost everywhere. It is routine to verify that M^* converges to the truth with stability. To show that M^* has a maximal domain of convergence, note that it converges to the truth in all states in $|\mathbf{Yes}|$ and in all states in $|\mathbf{No}|$ except the Cartesian scenarios of induction. No learning method converges to the truth in strictly more states. For to do so is to converge to the truth both in a normal state (\mathbf{Yes}, \vec{e}) and its Cartesian counterpart (\mathbf{No}, \vec{e}) , which is impossible. This establishes maximal convergence for M^* , and finishes the proof of the existential claim.

To establish the first part of the universal claim “never be counterinductive”, it suffices to invoke the proof that has already been detailed in appendix B.1, or simply to note that it is a corollary of theorem 7.3. Note that the proof relies only on the two modes of convergence to the truth, “almost everywhere” and “stable”. To establish the second part “be enumeratively inductive”, suppose that M is a learning method for the hard raven problem that converges to the truth on a maximal domain, and that M is never counterinductive (making use of the first part). It suffices to show that M is enumeratively inductive, as follows. Since M is never counterinductive, M fails to converge to the truth in each Cartesian scenario of induction. So the domain of convergence of M is included in that of M^* , which has been proved to be a maximal domain of convergence. But M converges on a maximal domain, so M must have the same domain of convergence as M^* . Then M converges to the truth in every state (\mathbf{Yes}, \vec{e}) contained in $|\mathbf{Yes}|$. It follows that M is enumeratively inductive. \square

Proof of Theorem 6.8. The proof of the existential claim is the crux, so let me first present the proof of the easy part, the universal claim. Just copy

the proof of the universal claim in theorem 6.5 (i.e. the preceding paragraph), paste it here, and apply the following replacements: First, replace the reference to theorem 7.3 by the reference to its Bayesian counterpart, theorem 7.5. Second, replace ‘learning method’ by ‘probabilistic prior’. Third, replace M by \mathbb{P} . As the last step, replace M^* by \mathbb{P}^* , which is the probabilistic prior to be constructed below for proving the existential claim.

To prove the existential claim, construct a witness \mathbb{P}^* as a linear combination of two other probabilistic priors:

$$\mathbb{P}^* = \frac{1}{2} \mathbb{P}_0 + \frac{1}{2} \mathbb{P}_1,$$

where \mathbb{P}_0 and \mathbb{P}_1 are defined as follows. Let \mathbb{P}_0 be the probability function generated by, so to speak, assuming that **Yes** is true and observations of $+$, 0 , $-$ are i.i.d. (independent and identically distributed) random variables, with equal probability $1/2$ for $+$ and for 0 , and with probability 0 for $-$. So:

$$\begin{aligned} \mathbb{P}_0(\mathbf{Yes}) &= 1. \\ \mathbb{P}_0(e_1, \dots, e_n) &= \begin{cases} \left(\frac{1}{2}\right)^n & \text{if } e_i \neq - \text{ for each } i \leq n, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Similarly, let \mathbb{P}_1 be the probability function generated by, so to speak, assuming that **No** is true and observations of $+$, 0 , $-$ are i.i.d. random variables with equal probability $1/3$ for $+$, for 0 , and for $-$. So:

$$\begin{aligned} \mathbb{P}_1(\mathbf{No}) &= 1. \\ \mathbb{P}_1(e_1, \dots, e_n) &= \left(\frac{1}{3}\right)^n. \end{aligned}$$

It suffices to show that \mathbb{P}^* , defined as the half-and-half mixture of \mathbb{P}_0 and \mathbb{P}_1 , converges to the truth with all the three modes mentioned in the existential claim. By the construction of \mathbb{P}^* , we have:

$$\begin{aligned} \mathbb{P}^*(\mathbf{Yes}) &= 1/2. \\ \mathbb{P}^*(\mathbf{No}) &= 1/2. \\ \mathbb{P}^*(e_1, \dots, e_n \mid \mathbf{Yes}) &= \mathbb{P}_0(e_1, \dots, e_n) = \begin{cases} \left(\frac{1}{2}\right)^n & \text{if } e_i \neq - \text{ for each } i \leq n, \\ 0 & \text{otherwise.} \end{cases} \\ \mathbb{P}^*(e_1, \dots, e_n \mid \mathbf{No}) &= \mathbb{P}_1(e_1, \dots, e_n) = \left(\frac{1}{3}\right)^n. \end{aligned}$$

Now, calculate conditional probability $\mathbb{P}^*(\mathbf{Yes} \mid e_1, \dots, e_n)$ by plugging the above probability values into the following instance of Bayes' theorem:

$$\begin{aligned} & \mathbb{P}^*(\mathbf{Yes} \mid e_1, \dots, e_n) \\ = & \frac{\mathbb{P}^*(e_1, \dots, e_n \mid \mathbf{Yes}) \mathbb{P}^*(\mathbf{Yes})}{\mathbb{P}^*(e_1, \dots, e_n \mid \mathbf{Yes}) \mathbb{P}^*(\mathbf{Yes}) + \mathbb{P}^*(e_1, \dots, e_n \mid \mathbf{No}) \mathbb{P}^*(\mathbf{No})}. \end{aligned}$$

Then we have:

$$\mathbb{P}^*(\mathbf{Yes} \mid e_1, \dots, e_n) = \begin{cases} \frac{1}{1+(2/3)^n} & \text{if } e_i \neq - \text{ for each } i \leq n, \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

$$\mathbb{P}^*(\mathbf{No} \mid e_1, \dots, e_n) = 1 - \mathbb{P}^*(\mathbf{Yes} \mid e_1, \dots, e_n). \quad (5)$$

$$\lim_{n \rightarrow \infty} \frac{1}{1 + (2/3)^n} = 1. \quad (6)$$

By the above three equations, (4)-(6), it follows that \mathbb{P}^* converges to the truth in all states in $|\mathbf{Yes}|$, and in all states in $|\mathbf{No}|$ except the Cartesian scenarios of induction. But recall that, by lemma 4.3, the set of the Cartesian scenarios of induction is negligible within the topological space $|\mathbf{No}|$. So \mathbb{P}^* converges to the truth almost everywhere. Argue for stable convergence by considering the following two cases.

Case (i): suppose that $\mathbb{P}^*(\mathbf{Yes} \mid e_1, \dots, e_n) > 1/2$ and state $s \in |\mathbf{Yes}| \cap |(e_1, \dots, e_n)|$. So $s = (\mathbf{Yes}, \vec{e})$, where \vec{e} is an infinite $+/0$ sequence. By equation (4) and the fact that $1/(1 + (2/3)^n)$ is a monotonically increasing function of n that converges to 1 as $n \rightarrow \infty$, we have: \mathbb{P}^* converges to the truth in s and $\mathbb{P}^*(\mathbf{Yes} \mid e_1, \dots, e_n) \geq \mathbb{P}^*(\mathbf{Yes} \mid e_1, \dots, e_n, \dots, e_{n'})$ for any $n' \geq n$.

Case (ii): suppose that $\mathbb{P}^*(\mathbf{No} \mid e_1, \dots, e_n) > 1/2$ and state $s \in |\mathbf{No}| \cap |(e_1, \dots, e_n)|$. So, by equations (4) and (5), (e_1, \dots, e_n) contains an occurrence of $-$. Then $s = (\mathbf{No}, \vec{e})$, where $e_i = -$ for some $i \leq n$. So $\mathbb{P}^*(\mathbf{No} \mid e_1, \dots, e_n, \dots, e_{n'}) = 1$ for all $n' \geq n$. So we have: \mathbb{P}^* converges to the truth in s and $\mathbb{P}^*(\mathbf{No} \mid e_1, \dots, e_n) \geq \mathbb{P}^*(\mathbf{No} \mid e_1, \dots, e_n, \dots, e_{n'})$ for any $n' \geq n$.

By the results of cases (i) and (ii), \mathbb{P}^* converges to the truth with stability. To establish maximal domain of convergence, suppose for *reductio* that there is a probability function \mathbb{P} that has a strictly more inclusive domain of convergence than \mathbb{P}^* does. But \mathbb{P}^* converges to the truth in all states except the Cartesian scenarios of induction. So \mathbb{P} must converge to the truth in a certain normal state and in its Cartesian counterpart, which is impossible. So \mathbb{P}^* converges to the truth on a maximal domain. \square

B.4 Proofs for Section 7: Ockham's Razor

The proofs presented in this section rely on the forcing lemma proved in section B.2.

Proof of Theorem 7.3. Suppose that learning method M converges to the truth almost everywhere for problem $(\mathcal{H}, \mathcal{E}, \mathcal{S})$. To prove the side $1 \Rightarrow 2$ by contraposition, suppose that M does not follow Ockham's tenacious razor. It suffices to show that M does not converge to the truth with stability. Discuss the following two exhaustive cases.

Case (i): Suppose that M violates the tenacity condition. That is, $M(e_1, \dots, e_n) = h$ and $M(e_1, \dots, e_n, \dots, e_{n'}) \neq h$, where $(e_1, \dots, e_n, \dots, e_{n'})$ is compatible with h . By that compatibility, choose a state s in the nonempty set $|h| \cap (e_1, \dots, e_n, \dots, e_{n'})$. It follows that, given stage n in state s , M outputs the truth h but it has not converged to the truth. So M fails to converge to the truth with stability.

Case (ii): Suppose that M violates Ockham's razor. Then $M(e_1, \dots, e_n) = h$, for some $(e_1, \dots, e_n) \in \mathcal{E}$ and some $h \in \mathcal{H}$, but there exists another hypothesis $h' \in \mathcal{H}$ that is compatible with (e_1, \dots, e_n) and simpler than h . Since (e_1, \dots, e_n) is compatible with h' , by the forcing lemma B.3 and the almost everywhere convergence of M , we have: (e_1, \dots, e_n) can be extended into a data sequence $(e_1, \dots, e_n, \dots, e_{n'}) \in \mathcal{E}$ such that, first, $M(e_1, \dots, e_n, \dots, e_{n'}) = h'$ and, second, $(e_1, \dots, e_n, \dots, e_{n'})$ is compatible with h' . Since $(e_1, \dots, e_n, \dots, e_{n'})$ is compatible with h' and since h' is simpler than h , it follows that $(e_1, \dots, e_n, \dots, e_{n'})$ is also compatible with h . By that compatibility, choose a state $s \in |h| \cap |(e_1, \dots, e_n, \dots, e_{n'})|$. So, given the earlier stage n in state s , M outputs the truth, h , but has not converged to the truth, for $M(e_1, \dots, e_n, \dots, e_{n'}) \neq h$. It follows that M fails to converge to the truth with stability.

To prove the side $2 \Rightarrow 1$, it suffices to show that the tenacity condition (alone) implies convergence to the truth with stability. Suppose that M has the tenacity property, and that M outputs the truth h given stage n in state $s = (h, \vec{e})$. It suffices to show that M has converged to the truth given the same stage n in the same state s . Note that, for any natural number i , the data sequence (e_1, \dots, e_{n+i}) extends (e_1, \dots, e_n) and is compatible with h . So, by the tenacity condition, $M(e_1, \dots, e_{n+i}) = h$, for all $i \geq 0$. It follows that M has converged to the truth, h , given stage n in state s . \square

Proof of Theorem 7.5. Copy the proof of the $1 \Rightarrow 2$ side of theorem 7.3, and paste it here. Then apply the following replacements. For case (i):

- First, replace $M(e_1, \dots, e_n) = h$ and $M(e_1, \dots, e_n, \dots, e_{n'}) \neq h$ by $1/2 < \mathbb{P}(h | e_1, \dots, e_n) > \mathbb{P}(h | e_1, \dots, e_n, \dots, e_{n'})$.

- And then replace each occurrence of M by \mathbb{P} .

For case (ii):

- First, replace $M(e_1, \dots, e_n) = h$
by $\mathbb{P}(h \mid e_1, \dots, e_n) > 1/2$.
- Then replace $M(e_1, \dots, e_n, \dots, e_{n'}) = h'$
by $\mathbb{P}(h' \mid e_1, \dots, e_n, \dots, e_{n'}) > 1/2$.
- Then replace $M(e_1, \dots, e_n, \dots, e_{n'}) \neq h$
by $\mathbb{P}(h \mid e_1, \dots, e_n, \dots, e_{n'}) \not> 1/2$
- And, as the last step, replace each occurrence of M by \mathbb{P} .

This finishes the proof of the $1 \Rightarrow 2$ side.

To prove that the converse $2 \Rightarrow 1$ does *not* hold, construct a problem $\mathcal{P} = (\mathcal{H}, \mathcal{E}, \mathcal{S})$ as follows. Consider only the following data streams, where m and n are arbitrary natural numbers:

$$\begin{aligned} s_\omega &= 0^\omega. \\ s_m &= 0^m 1^\omega. \\ s_{mn} &= 0^m 1^n 2^\omega. \end{aligned}$$

Their initial segments form the evidence space \mathcal{E} . The hypothesis space \mathcal{H} consists of

- $h =$ “The actual sequence will not end with occurrences of 2.”
- $h' =$ “It will.”

The state space \mathcal{S} consists of (h, s_ω) , (h, s_m) , and (h', s_{mn}) , for all natural numbers m and n . Construct a countably additive probability function \mathbb{P} that assigns the following probabilities to singletons of states:

$$\begin{aligned} \mathbb{P}\{s_\omega\} &= 0. \\ \mathbb{P}\{s_m\} &= \left(\frac{1}{2}\right)^{m+1} \times 60\%. \\ \mathbb{P}\{s_{mn}\} &= \left(\frac{1}{2}\right)^{m+1} \times 40\% \times \left(\frac{1}{2}\right)^{n+1}. \end{aligned}$$

Those assignments of probabilities are designed to ensure the following:

$$\begin{aligned}\mathbb{P}\{s_m\} &= \left(\frac{1}{2}\right)^{m+1} \times 60\%. \\ \mathbb{P}\{s_{m0}, s_{m1}, s_{m2}, \dots\} &= \left(\frac{1}{2}\right)^{m+1} \times 40\%. \\ \mathbb{P}\{s_m, s_{m0}, s_{m1}, s_{m2}, \dots\} &= \left(\frac{1}{2}\right)^{m+1}. \\ \sum_{m=0}^{\infty} \mathbb{P}\{s_m, s_{m0}, s_{m1}, s_{m2}, \dots\} &= 1.\end{aligned}$$

It follows that, for each natural number m , we have:

$$\mathbb{P}(h | 0^m) = 60\%.$$

So \mathbb{P} fails to converge to the truth H in state 0^ω . It is routine to verify that \mathbb{P} converges to the truth in all the other states and, hence, does so almost everywhere. It is also routine to verify that \mathbb{P} follows Ockham's tenacious razor. In state (h, s_ω) and given information 0^m , \mathbb{P} assigns a probability greater than $1/2$ (namely 60%) to the truth (namely h) but fails to have started to stably converge to the truth, because it even fails to converge to the truth in that state. So \mathbb{P} fails to converge to the truth with stability. This finishes the proof. \square